

High-Momenta Estimates for the Klein-Gordon Equation: Long-Range Magnetic Potentials and Time-Dependent Inverse Scattering ^{*}

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Abstract

The study of obstacle scattering for the Klein-Gordon equation in the presence of long-range magnetic potentials is addressed. Previous results of the authors are extended to the long-range case and the results the authors previously proved for high-momenta long-range scattering for the Schrödinger equation are brought to the relativistic scenario. It is shown that there are important differences between relativistic and non-relativistic scattering concerning long-range. In particular, it is proved that the electric potential can be recovered without assuming the knowledge of the long-range part of the magnetic potential, which has to be supposed in the non-relativistic case. The electric potential and the magnetic field are recovered from the high momenta limit of the scattering operator, as well as fluxes modulo 2π around handles of the obstacle. Moreover, it is proved that, for every $\hat{\mathbf{v}} \in \mathbb{S}^2$, $A_\infty(\hat{\mathbf{v}}) + A_\infty(-\hat{\mathbf{v}})$ can be reconstructed, where A_∞ is the long-range part of the magnetic potential. A simple formula for the high momenta limit of the scattering operator is given, in terms of magnetic fluxes over handles of the obstacle and long-range magnetic fluxes at infinity, that are introduced in this paper. The appearance of these long-range magnetic fluxes is a new effect in scattering theory.

1 Introduction

1.1 The Aharonov-Bohm Effect : Essential Features

The Aharonov-Bohm effect is a fundamental issue in physics. It describes a phenomenon that is not compatible with classical mechanics, but that can be predicted from quantum physics. Moreover, it describes the physically significant (classical) electromagnetic quantities in quantum mechanics. It is, thus, not only a specific phenomenon but a highly influential concept in quantum theory, and its experimental verification is an important confirmation of its accuracy as a physical theory. We now describe in more detail what we just mentioned.

The situation we analyse is an electron (a test charged particle) in the presence of a classical magnetic field

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(although in this paper we consider also electric fields, now take for the moment only into account a magnetic field). According to classical mechanics (Newton's law) the motion of the particle is totally determined by the force (acting in the particle) and the initial position and velocity. In this case the force is given by the Lorentz formula:

$$q\mathbf{v} \times B, \quad (1.1)$$

where \mathbf{v} is the velocity of the particle, q is the charge and B is the magnetic field (evaluated at the position of the particle). This gives, as classical mechanics predicts, the complete picture of the electron motion. Therefore, the only physical quantity affecting the behaviour of the particle is the magnetic field. However, the world is not so simple and what classical mechanics predicts is not correct. There are other physically significant quantities besides the force. These quantities are not easy to precise, but let's discuss a little bit more about it (in the text below we consider how to determine them):

In quantum physics the situation we just described can be modeled through the Schrödinger equation

$$i\frac{\partial}{\partial t}\phi = (\mathbf{p} - A)^2\phi, \quad (1.2)$$

where $\mathbf{p} = -i\nabla$ is the momentum operator, A is a magnetic potential such that $\nabla \times A = B$. Here we set \hbar to one, the mass of the particle equal 1/2, and include in A the electric charge. After a close look at the Schrödinger equation one might think that the physical significant quantity we are looking for is the magnetic potential. This, nevertheless, is not correct. The reason is that in quantum mechanics the wave function ϕ is not uniquely determined, but depends on a representation, i.e., it is defined up to a unitary transformation, like a gauge transformation. Then, changing A by $A + \nabla\lambda$, for some scalar function λ , must not change any physical prediction. Thus, describing physically significant quantities is subtle business. Here is where the contribution of Aharonov and Bohm [2] takes place (see also [17]): They consider an infinitely long straight thin solenoid and a magnetic field confined to it. They describe a situation in which an electron wave packet consisting in two separated beams is directed to the solenoid. Each beam passes through different sides of the solenoid and they are brought together behind the solenoid in order to produce an interference pattern. It turns out that the interference pattern depends on the magnetic field enclosed in the solenoid, even if the electron never touches it. Let's use loosely the Schrödinger equation to explain this issue (for a rigorous justification see [6]). Let us suppose that the solenoid is located at the vertical axis and that the whole situation does not depend on the vertical variable. Then, we reduce the problem to two dimensions. Since the magnetic field vanishes outside the solenoid, the magnetic potential is gauge-equivalent to zero in every connected region on its exterior, but not in the full exterior of the solenoid.

When the magnetic field inside the solenoid is zero, the solution to the Schrödinger equation (1.2) is supposed to consist of two beams,

$$\phi_0(t, x) = \phi_{0,1}(t, x) + \phi_{0,2}(t, x),$$

where $\phi_{0,j}$, $j = 1, 2$ are separately solutions to the Schrödinger equation and they remain in connected regions away from the solenoid. Furthermore, as time increases, $\phi_{0,1}$ is supposed to pass through the left of the solenoid and $\phi_{0,2}$

to the right of it. Aharonov and Bohm argued [2] that when the magnetic field inside the solenoid is not zero the solution to the Schrödinger equation is given again by two beams,

$$\phi(t, x) = \phi_1(t, x) + \phi_2(t, x), \text{ where } \phi_j(t, x) = e^{-i\lambda_j(x)} \phi_{0,j}(t, x), j = 1, 2,$$

and λ_j is the circulation $\int A \cdot dx$ of the magnetic potential along the path of the beam $\phi_{0,j}, j = 1, 2$. Furthermore, assuming that initially (at $t = 0$) both beams are close to each other and located far from the solenoid near a point $x_{-\infty}$ we can take the functions λ_j as follows,

$$\lambda_j(x) := \int_{C_j(x_{-\infty}, x)} A \cdot dx, \quad (1.3)$$

where $C_j(x_{-\infty}, x)$ is a simple differentiable path joining the points $x_{-\infty}$ and x , where for $j = 1, 2$ the path goes, respectively, to the left and to the right of the solenoid. The two beams are brought together at some point, x_{∞} , behind the solenoid, where ϕ_1 acquires a phase $e^{-i \int_{C_1(x_{-\infty}, x_{\infty})} A \cdot dx}$ and ϕ_2 acquires a phase $e^{-i \int_{C_2(x_{-\infty}, x_{\infty})} A \cdot dx}$.

We denote by \mathcal{C} the simple closed curve obtained joining $C_1(x_{-\infty}, x_{\infty})$ and $C_2(x_{-\infty}, x_{\infty})$ with counterclockwise orientation. Then, the difference in phase between the left and the right beams at the point x_{∞} is given by (we use Stokes' theorem)

$$e^{-i \int_{\mathcal{C}} A} = e^{-i \Phi_B}, \quad (1.4)$$

where Φ_B is the magnetic flux in a transverse section of solenoid. The approximate solution described above is the prediction of Aharonov and Bohm. In [6] we called it the Ansatz of Aharonov and Bohm. Note that it consists of multiplying the free solution (when the magnetic field inside the solenoid is zero) by the Dirac magnetic factor [11]. In [6] we proved rigorously that the Aharonov-Bohm Ansatz is indeed an approximate solution and we gave error bounds. Actually, in [6] we considered the case of a toroidal magnet, as in the works where the Aharonov-Bohm prediction was experimentally verified [26, 27, 28, 29].

As pointed out by Aharonov and Bohm, the phase factor (1.4) can be predicted from quantum mechanics, even though the particles never touch the solenoid, i.e., the force produced by the magnetic field at the position of the particle is zero, all the time. This can be interpreted in the following two ways:

1. The magnetic field acts non locally.
2. Some properties of the magnetic potential are physically significant.

Regardless which interpretation we choose, it is important to precise the quantities determining the physics of the problem. Apparently, they are the electromagnetic fields and the fluxes of the magnetic potential, modulo 2π , over closed paths. This is in agreement with the discussion above, the complete description of electrodynamics in terms of non-integrable factors (see [34] and [11]) and the experimental results in [26, 27, 28, 29].

1.1.1 Relevance of High Velocity and Relativistic Scattering

In the explanation above we assumed that we can control the fate of the beams ϕ_1 and ϕ_2 and that they stay in a connected region not touching the solenoid all the time. This is not possible to achieve, but only approximately. However, the accuracy of the approximation depends on how ballistic is the motion of each beam, in order to control the spreading of the beams. Then, have to assume high enough momenta. The magnitude of the momenta, that we have to choose in order to have a good approximation, depends on the particular geometry and physical parameters of the system. This is why a relativistic theory (allowing relatively high momenta) is relevant.

1.2 Historical Context and the Necessity of Toridal Geometries

There is very a large literature on the Aharonov-Bohm effect. We, of course, do not pretend to be exhaustive, but to report the main advances in relation to our work. For an extensive review up to 1989 see [22] and [23]. We give here more recent references, but only the key contribution regarding our work. The readers are advised to look at the literature of our references, if they are searching for a complete record.

The two dimensional model of Aharonov and Bohm (see [2]) has the disadvantage of requiring infinite straight solenoids (see the discussion in Section 1.1). They, of course, do not exist in nature. The argument that sufficiently long (straight) solenoids could be considered infinite is controversial for the following reason: The topology of the exterior of a finite solenoid is trivial in the sense that all closed curves can be continuously deformed to a point. This implies, from Stoke's theorem, that the magnetic field cannot be confined in the solenoid and, therefore, the field leakage produces a flux that equals the magnetic flux inside the solenoid. Moreover, scattering experiments regularly send and detect particles from faraway of the target. In the description of the Aharonov-Bohm prediction we did above (Section 1.1) we used a wave packed separated in two beams. The beams are sent and detected faraway from the solenoid. Even though the magnetic field is very weak, the magnetic flux outside the magnet enclosed by the trajectory of both beams could be of the same order than the flux enclosed in the magnet, since the beams travel long distances. Then, magnetic flux enclosed by the paths the electrons follow could be significantly different from the flux in the solenoid itself. This is a situation when long enough might not signify infinite, because the topology of the exterior of long enough solenoids and the topology of the exterior of an infinite solenoid are dramatically different. Another reason why the infinite solenoid scenario might be problematic (and this is what we analyse and prove here and in [7]) is that in two dimensions the magnetic potentials must be long-range and the long-range potentials influence scattering (the scattering operator) in a way that some information non related to fields or magnetic fluxes modulo 2π can be inferred from the scattering operator (this is already present in [2]). We believe that this information might be non-physical as we explained in the previous section.

The amount of papers, books and experiments dealing with the case of a straight solenoid (as proposed in [2])

is large. However, scientists recognized the problem of the field leakage already decades ago. Since then, the issue became controversial and a new geometry proposal emerged: The toroidal geometry. This geometry allows to confine a magnetic field without leakage (notice that the topology of the exterior of a toroidal magnet is not trivial in the sense that there are closed curves that cannot be continuously deformed to a point). This led to the seminal experiments, with toroidal magnets, carried out by Tonomura et al. [26, 27, 28, 29]. In these remarkable experiments they split a coherent electron wave packet into two parts. One travelled inside the hole of the magnet and the other outside the magnet. They brought both parts together behind the magnet and they measured the phase shift produced by the magnetic flux enclosed in it, giving a strong evidence of the existence of the Aharonov-Bohm effect.

The experiments of Tonomura et al. [26, 27, 28, 29] reduced the controversy to a lower scale. The interpretation of the results was the new trend for some scientists. Some works proposed an interpretation in which the results by Tonomura et al. could be explained by the action of a force. See, for example, [9, 19] and the references quoted there. The force they referred to would accelerate the electron producing a time delay. In a recent experiment Caprez et al. [10] obtained that there is no acceleration. Then, they proved experimentally that the explanation of the results of the Tonomura et al. experiments by the action of a force is wrong.

From the theoretical point of view some efforts have been directed to justify that the Aharonov-Bohm Ansatz approximates correctly the solution to the Schrödinger equation. There have been numerous works trying to provide such approximations. Several Ansätze have been proposed, without giving error bound estimates. Most of these attempts are qualitative, although some of them give numerical values. Fraunhofer diffraction, first-order Born and high-energy approximations, Feynman path integrals and the Kirchhoff method in optics were used. For a review of the literature up to 1989 see [22] and [23]. Recently we rigorously proved that the Ansatz of Aharonov and Bohm is a good approximation to the solution of the Schrödinger equation and we analysed the full scattering picture, see [4, 5, 6]. In particular, in [5] we gave a rigorous quantitative proof, under the experimental conditions, that quantum mechanics predicts the experimental results of Tonomura et al. [26, 27, 28, 29]. In this work and in [8] we address the relativistic case. This paper is dedicated to long-range potentials, whereas [8] considers short-range potentials.

1.2.1 Relevance of Long-Range Potentials

In the proposal of Aharonov and Bohm (see [2]), the infinite straight solenoids restrict the space to two dimensions (see Section 1.1). However, the two dimensional situation requires long-range magnetic potentials. Through these potentials the scattering operator encodes information that is not related to fluxes modulo 2π nor to the electromagnetic field. If this was a physical information, we would have new phenomena. However, we believe that this information is not physical, see Section 1.1. The question at stake is: Which physical information can be extracted from the scattering operator ? i.e. to what extent can we rely on the scattering operator ?. Notice that the Aharonov-Bohm effect gives an example in which the scattering cross section is not the only information we can extract from scattering. The analysis of this in two dimensions for the non-relativistic case is done in [7]. Here we address the question for the

relativistic scenario. Notice that there are important differences between both cases, as we explain in this text.

1.3 The Role of Long-Range Magnetic Potentials – Description of Our Model and Further Historical Context

We study obstacle scattering of charged relativistic particles in the presence of long-range magnetic potentials. The obstacle K is assumed to be a finite union of handle bodies, for example of tori and balls. Inside it there is an inaccessible magnetic field. In particular, we focus on the effects of the long-range part of the magnetic potentials in high-momenta scattering. This article extends the results in [8], where only short-range magnetic potentials are addressed, and proves in the relativistic case results similar to the ones in [7], that considers the Schrödinger equation. We prove that all results for the Klein-Gordon equation in [8] are valid in the long-range case, but furthermore, we demonstrate that some information from the long-range part of the magnetic potentials can be reconstructed from high-momenta scattering. The role of long-range magnetic potentials in inverse-scattering has lately acquired interest (see [7], [14], [15] and [16]). The question at stake is: What are the properties of the magnetic potentials that can be recovered from the scattering operator ? This is a subtle question because the magnetic potentials are not physically significant in classical physics. Therefore, the above question is of purely quantum mechanical nature. Moreover, according to the complete description of electromagnetism in terms of non-integrable phase factors introduced in [34] (see also [11]) and the experimental results on the Aharonov-Bohm effect (see [10], and [26]-[29]), the only observable quantities (related to the magnetic potential in our setting) are magnetic fluxes modulo 2π over the handles of the obstacle. Nevertheless, it is proved in [7] and [14]-[15] that the long-range part of the magnetic potential (that is not related to the magnetic field or magnetic fluxes around handles) can be recovered from the (non-relativistic) scattering operator, in certain situations. In this paper we go further and prove similar results for relativistic equations, more precisely the Klein-Gordon equation. This brings new insights to the understanding of long-range magnetic effects in quantum mechanics, because differences and similarities, with respect to the non-relativistic case, appear. For example, it is shown in [7] that in the high-velocity limit of the (non-relativistic) scattering operator the long-range part of the magnetic potential and the electric potential are coupled, which implies that we have to assume the knowledge of the long-range part of the magnetic potential in order to be able to recover the electric potential and the other way around (we have to assume the knowledge of the electric potential in order to recover the long-range part of the magnetic potential). This also happens in [14]-[15] (see the discussion about this fact in the introduction of [7]). Fortunately, this problem seems to be artificial because considering special relativity in our equations (using the Klein-Gordon equation) decouples the electric and the magnetic potentials in the high-momenta limit. Then, the high momenta limit of the scattering operator permits the reconstruction of the electric potential without requiring the knowledge of the magnetic potential, which is one of the results in this paper. Additionally, in contrast to the non-relativistic case, in the relativistic situation, the fluxes around the handles of the obstacle can be recovered modulo 2π only if the electric potential (and the magnetic field) vanish, otherwise we can only recover them modulo π . This is physically

reasonable, because of the relativistic duality between the electric and the magnetic fields. Recovering the magnetic field from high momenta scattering does not distinguish between relativistic and non-relativistic models, as we show in this text. Denoting by $A_\infty(\hat{\mathbf{v}})$, $\hat{\mathbf{v}} \in \mathbb{S}^2$, the long-range part of the magnetic potential (see Proposition 6.2), relativistic scattering only allows us recovering $A_\infty(\hat{\mathbf{v}}) + A_\infty(-\hat{\mathbf{v}})$ and not $A_\infty(\hat{\mathbf{v}})$, $\hat{\mathbf{v}} \in \mathbb{S}^2$, as is the case for the non-relativistic Schrödinger equation.

Finally, we give a simple formula for the high momenta limit of the scattering operator in terms of magnetic fluxes over handles of the obstacle and long-range magnetic fluxes at infinity, that we introduce in this paper. The appearance of these long-range magnetic fluxes is a new effect in scattering theory. This result is also true in the non-relativistic case of the Schrödinger equation in three dimensions considered in [4], with a similar proof, and also in the two dimensional case studied in [7], with the necessary changes due to the differences in the geometry.

The Aharonov-Bohm effect [2], [17] for non-relativistic equations and short-range magnetic potentials is studied, for example, in [4], [5], [6], [14], [15], [16], [21] and [30], and the references quoted there. For high-momenta scattering for relativistic equations in the whole space, see [12] and [20]. The magnetic Schrödinger equation, in the whole space, is studied in [3]. The time dependent methods for inverse scattering that we use are introduced in [13], for the Schrödinger equation. A survey about many different applications of this time dependent method for inverse scattering can be found in [33]. The direct scattering problem for the Klein-Gordon equation is studied in [18], [31] and [32] and the references cited there.

1.4 Main Results and Description of the Paper

Here we describe our main results and give a short guideline of the paper. The obstacle is properly defined in Section 2.2.1. It consists in a union of handle bodies. In Figure 1 we draw an example of the class of obstacles we consider. For each handle of K (let's say the handle number j), we choose a curve $\hat{\gamma}_j$ surrounding it. The classes of magnetic potentials we use depend on the magnetic fluxes over the curves $\hat{\gamma}_j$, $j \in \{1, \dots, m\}$. We define in Section 2.1 three classes of magnetic potentials (see Definitions 2.4 and 2.7): The most general one is denoted by $\mathcal{A}_\Phi^{\text{LR}}(B)$ (the superscript LR stands for long-range), $\mathcal{A}_\Phi^{\text{SR}}(B)$ denotes the class of short-range magnetic potentials and, finally, we define a class of regular long-range magnetic potentials $\mathcal{A}_{\Phi,\delta}(B)$. Here B , defined in the same section, is the magnetic field outside the obstacle and Φ represents the circulations of the magnetic potentials over the curves $\hat{\gamma}_j$, $j \in \{1, \dots, m\}$. We denote by A_0 the electric potential, it is defined in Section 2.1. As in [8], [31], [32] we write the free and the interacting Klein-Gordon equations as first order in time 2 by 2 systems, respectively, in the free Hilbert space \mathcal{H}_0 and in the interacting Hilbert space $\mathcal{H}(\underline{A})$. See Section 2.3.1. The free Hamiltonian, H_0 , is introduced in (2.14). The perturbed Hamiltonian, $H(\underline{A})$, is defined in (2.21). Here $\underline{A} = (A_0, A)$, A being the magnetic potential. In Section 3 we prove the existence of the wave and scattering operators. The wave operators are defined by the strong limits

$$W_\pm(\underline{A}) = \text{s-}\lim_{t \rightarrow \pm\infty} e^{itH(\underline{A})} J e^{-itH_0}, \quad (1.5)$$

where J is a bounded identification operator from \mathcal{H}_0 into $\mathcal{H}(\underline{A})$. The scattering operator is

$$S(\underline{A}) = W_+(\underline{A})^* W_-(\underline{A}).$$

One of our main results is Theorem 4.1 in which we prove (with error bounds) that the high momenta limit of the scattering operator, in the representation where the free Klein-Gordon operator is diagonal (see (2.17)), is given by

$$\begin{pmatrix} e^{i \int_{-\infty}^{\infty} dr (A \cdot \nu - A_0)(\mathbf{x} + r\nu)} & 0 \\ 0 & e^{-i \int_{-\infty}^{\infty} dr (A \cdot \nu + A_0)(\mathbf{x} + r\nu)} \end{pmatrix}, \quad (1.6)$$

where \mathbf{x} is the multiplication operator with respect to the variable $x \in \mathbb{R}^3$. which extends Theorem 2.8 in [8] to the long-range case. Eq. (1.6) is used in Section 5 to recover the electric potential and the magnetic field in Theorem 5.2 and magnetic fluxes over the handles of the obstacle, modulo 2π , in Theorem 5.3. Our main results on long-range effects in high-momenta scattering are presented in Section 6, in particular in Theorems 6.5 and 6.7. For every magnetic potential $A \in \mathcal{A}_{\Phi, \delta}(B)$, $\delta > 1$, we define its long-range part by

$$A_{\infty}(\hat{\mathbf{v}}) = \lim_{s \rightarrow \infty} s A_{\infty}(s \hat{\mathbf{v}}).$$

In Theorem 6.5 we prove that $A_{\infty}(\hat{\mathbf{v}}) + A_{\infty}(-\hat{\mathbf{v}})$, for every $\hat{\mathbf{v}} \in \mathbb{S}^2$, can be recovered from the high momenta limit of the scattering operator. In Theorem 6.7 we give a simple formula for the high momenta limit of the scattering operator, assuming the electromagnetic field vanishes, in terms of magnetic fluxes modulo 2π around the handles of the obstacle and a long-range magnetic fluxes at infinity that we introduce in Definition 6.1. As mentioned above, this result is also true in the non-relativistic case of the Schrödinger equation in three dimensions considered in [4], with a similar proof, and also in the two dimensional case studied in [7], with the necessary changes due to the differences in the geometry.

2 Model

2.1 Description of the Model

We extend the model introduced in [8] to the study of long-range magnetic potentials. In this section, several definitions and notations are borrowed from [8]. We briefly repeat some of them, for the convenience of the reader, and include the new ingredient: The long-range magnetic potentials. We refer the reader to [8] for a more detailed presentation.

2.2 General Notation

For every normed vector space X we denote by $\|\cdot\|_X$ its norm. If no confusion arises, we omit the subscript in case $X = L^2(O)$ or $L^2(O) \oplus L^2(O)$, for some open subset O in \mathbb{R}^3 , or X is a space of operators. We assume the same convention for inner products. In this text we denote by $B(x; r)$ the open ball in \mathbb{R}^3 centred at x and with radius r . For any vector $\mathbf{v} \in \mathbb{R}^3 \setminus \{0\}$ we designate, $\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$. For any set $O \subset \mathbb{R}^3$ we denote by O^c its complement, by O^o its

interior, and by χ_O the characteristic function of O . By C we denote a positive, non-specified, constant. We use the standard notation $\langle x \rangle = (1 + x^2)^{1/2}$, for every $x \in \mathbb{R}^3$. The Schwartz space of rapidly decreasing C^∞ -functions in \mathbb{R}^3 is denoted by $\mathcal{S}(\mathbb{R}^3)$. For every $n \in \mathbb{N}$ and every open set O in \mathbb{R}^3 , we denote by $\mathbf{H}^n(O)$ the Sobolev space of functions with distributional derivatives up to order n square integrable, and by $\mathbf{H}_0^n(O)$ the closure of $C_0^\infty(O)$ in $\mathbf{H}^n(O)$, see [1]. For every strictly positive function $\omega : O \mapsto \mathbb{R}$ we denote by $\mathbf{H}_\omega^n(O)$ the corresponding weighted Sobolev space. For every $\phi \in \mathbf{H}_\omega^n(O)$,

$$\|\phi\|_{\mathbf{H}_\omega^n(O)} := \sum_{\alpha_1 + \alpha_2 + \alpha_3 \leq n} \left\| \omega^{1/2} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \frac{\partial^{\alpha_3}}{\partial x_3^{\alpha_3}} \phi \right\|_{L^2(O)}.$$

We analyze charged relativistic particles moving outside a compact obstacle, K , in three dimensions. We denote by $\Lambda := \mathbb{R}^3 \setminus K$. Inside K there is an inaccessible magnetic field and in Λ there is an electromagnetic field. We denote by A_0 the electric potential and by B the magnetic field.

We recall the notation we choose for the momentum operator and the position operator. The momentum operator is denoted by

$$\mathbf{p} = -i\nabla$$

and the position operator is denoted by \mathbf{x} , which is the multiplication operator by the variable $x \in \mathbb{R}^3$. The momentum operator is, clearly, the multiplication operator by the variable $p \in \mathbb{R}^3$, in momentum space representation. We utilize along the paper \mathbf{v} to denote a fixed vector in \mathbb{R}^3 and $\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ (if $\mathbf{v} \neq 0$). We use ν to denote a fixed element of the sphere \mathbb{S}^2 and v will be positive real number, generally the norm of \mathbf{v} . We do not adopt the convention that a bold face symbol always represent a vector.

2.2.1 The Obstacle K

The obstacle K is a compact submanifold of \mathbb{R}^3 . We denote by $\{K_j\}_{j=1}^L$ its connected components. We assume that the K_j 's are handle bodies. For more details see [4], where the same obstacle is addressed. See Figure 1.

2.2.2 The Electromagnetic Field and the Potentials

DEFINITION 2.1 (The Magnetic Field). The magnetic field, B , is a real-valued, closed and bounded 2-form defined in $\bar{\Lambda}$. We assume that it is two times continuously differentiable. We additionally suppose that

$$\int_{\partial K_j} B = 0, \quad j \in \{1, 2, \dots, L\}, \quad (2.1)$$

and

$$\left| \left(\frac{\partial}{\partial x_1} \right)^a \left(\frac{\partial}{\partial x_2} \right)^b \left(\frac{\partial}{\partial x_3} \right)^c B(x) \right| \leq C(1 + |x|)^{-\mu}, \text{ for some } \mu > 2, \text{ and every } a, b, c \in \{0, 1, 2\} \text{ with } a + b + c \leq 2. \quad (2.2)$$

DEFINITION 2.2 (Electric Potential). The electric potential is a real-valued function, A_0 , satisfying (for some $\varepsilon > 0$)

$$\langle A_0^2 \phi, \phi \rangle \leq \langle -\Delta \phi, \phi \rangle + (m^2 - \varepsilon) \langle \phi, \phi \rangle, \quad (2.3)$$

for every $\phi \in \mathbf{H}_0^1(\Lambda)$. We, furthermore, assume that for some $\delta < 1/5$ there exists a constant C_δ such that

$$\langle A_0^2 \phi, \phi \rangle \leq \delta \langle -\Delta \phi, \phi \rangle + C_\delta \langle \phi, \phi \rangle, \quad (2.4)$$

for every $\phi \in \mathbf{H}_0^1(\Lambda)$. We assume, additionally, that for some C^∞ function κ , defined in \mathbb{R}^3 , such that $\kappa = 0$ in a neighborhood of K and with $1 - \kappa$ compactly supported, κA_0 is two times continuously differentiable and

$$\left| \left(\frac{\partial}{\partial x_1} \right)^a \left(\frac{\partial}{\partial x_2} \right)^b \left(\frac{\partial}{\partial x_3} \right)^c \kappa A_0(x) \right| \leq C(1 + |x|)^{-\zeta}, \text{ for some } \zeta > 1, \text{ and every } a, b, c \in \{0, 1, 2\} \text{ with } a + b + c \leq 2. \quad (2.5)$$

For sufficient conditions in order that (2.3) and (2.4) hold see [25], [30], [31].

2.3 Classes of Magnetic Potentials

We adopt the definitions, and notations, presented in Eq. (2.6) in [4] : We set $\{\hat{\gamma}_j\}_{j=1}^L$ the curves described in Figure 1.

DEFINITION 2.3. We denote by Φ a function $\Phi : \{\hat{\gamma}_j\}_{j=1}^L \rightarrow \mathbb{R}$. We call it the flux.

DEFINITION 2.4. We denote by $\mathcal{A}_\Phi^{\text{LR}}(B)$ the class of continuous 1- forms, A , defined in $\bar{\Lambda}$ such that $dA = B$, $\int_{\hat{\gamma}_j} A = \Phi(\hat{\gamma}_j)$, $\forall j \in \{1, 2, \dots, L\}$, and

$$|A(x)| \leq C(1 + |x|)^{-1}, \quad a(r) := \sup_{x \in \bar{\Lambda}, |x| \geq r} A(x) \cdot \frac{x}{|x|} \in L^1(\mathbb{R}). \quad (2.6)$$

If additionally

$$A(x) \leq C(1 + |x|)^{-\zeta}, \quad (2.7)$$

for some $\zeta > 1$, we say that $A \in \mathcal{A}_\Phi^{\text{SR}}(B)$. Here the superscript LR stands for long-range and the superscript SR stands for short-range.

REMARK 2.5. In Theorem 3.7 of [4] (see also Remark 2.4 of [8]) the Coulomb potential is constructed. This potential belongs to $\mathcal{A}_\Phi^{\text{SR}}(B)$ for some $\zeta > 1$ that depends on μ .

DEFINITION 2.6. For every $a, b \in [0, \infty)$ with $a + b > 2$, we define the function $\iota_{a,b} : \mathbb{R}^3 \rightarrow \mathbb{R}$:

$$\iota_{a,b}(x) := \begin{cases} \frac{1}{(1+|x|)^{\min(a,b)}} + \frac{1}{(1+|x|)^{a+b-2}}, & \text{if } a, b \neq 2, \\ \frac{1}{(1+|x|)^2} + \frac{\ln(e+|x|)}{(1+|x|)^{a+b-2}}, & \text{if } a = 2 \text{ or } b = 2. \end{cases} \quad (2.8)$$

DEFINITION 2.7 (Second Class of Long-Range Magnetic Potentials). For every vector potential $A \in \mathcal{A}_\Phi^{\text{LR}}(B)$, we designate by $\alpha_A : \Lambda \rightarrow \mathbb{R}$ the function $\alpha_A(x) := A(x) \cdot x, \forall x \in \Lambda$. Let $\delta > 1$. We denote by $\mathcal{A}_{\Phi,\delta}(B)$ the set of vector potentials $A \in \mathcal{A}_\Phi^{\text{LR}}(B) \cap C^2(\Lambda, \mathbb{R}^3)$ such that there is a constant C satisfying

$$\sum_{i=1}^3 \left| \frac{\partial}{\partial x_i} A(x) \right| \leq C \frac{1}{(1+|x|)^2}, \quad |\nabla \alpha_A(x)| \leq C \iota_{2,\delta}(x), \quad |\alpha_A(x)| \leq C \iota_{1,\delta}(x), \quad \left| \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \alpha_A(x) \right| \leq C \min \left(\iota_{3,\delta}(x), \frac{\ln(e+|x|)}{(1+|x|)^2} \right),$$

for all $x \in \Lambda, i, j \in \{1, 2, 3\}$.

It is not difficult to see (see Lemma 3.8 in [4]) that for any pair $A^{(1)}, A^{(2)} \in \mathcal{A}_\Phi^{\text{LR}}(B)$ there exists a C^1 0–form λ in $\bar{\Lambda}$ such that $A^{(2)} - A^{(1)} = \nabla \lambda$. λ is given by the formula

$$\lambda(x) := \int_{C(x_0, x)} (A^{(2)} - A^{(1)}), \quad (2.9)$$

for a fixed point x_0 in Λ and a C^∞ -curve $C(x_0, x)$ in Λ with starting point x_0 and ending point x . Moreover the limit

$$\lambda_\infty(x) := \lim_{r \rightarrow \infty} \lambda(rx) \quad (2.10)$$

exists and defines a continuous and homogeneous (of order zero) function in $\mathbb{R}^3 \setminus \{0\}$. Furthermore,

$$|\lambda_\infty(x) - \lambda(x)| \leq \int_{|x|}^\infty b(|x|), \text{ for some positive function } b \in L^1(0, \infty). \quad (2.11)$$

2.3.1 The Hamiltonians

As in [8], [31], [32] we write the Klein-Gordon equation as a 2 by 2 system that is first order in time, in order that it is a Hamiltonian equation.

2.4 Free Hamiltonian

The free Klein-Gordon equation is given by

$$\left(i \frac{\partial}{\partial t}\right)^2 \phi = (\mathbf{p}^2 + m^2) \phi, \quad (2.12)$$

where $\mathbf{p} := -i\nabla$ is the momentum operator and $m > 0$ is the mass of the particle, and the solution ϕ is a complex valued function defined in $\mathbb{R} \times \mathbb{R}^3$. For the free evolution the electromagnetic potentials are zero and there is no obstacle.

We define the operator $B_0 := (\mathbf{p}^2 + m^2)^{1/2}$, with domain the Sobolev space $\mathbf{H}^1(\mathbb{R}^3)$. Denote by \mathcal{H}_0 the Hilbert space

$$\mathcal{H}_0 := \text{dom}(B_0) \oplus L^2(\mathbb{R}^3), \quad (2.13)$$

with inner product $\langle \phi, \psi \rangle_{\mathcal{H}_0} := \langle B_0 \phi_1, B_0 \psi_1 \rangle + \langle \phi_2, \psi_2 \rangle$, for $\phi = (\phi_1, \phi_2)$, $\psi = (\psi_1, \psi_2)$. Note that the inner product of \mathcal{H}_0 is the sesquilinear form associated to the classical field energy of the free Klein-Gordon equation.

The free Hamiltonian is the self-adjoint operator

$$H_0 := \begin{pmatrix} 0 & i \\ -iB_0^2 & 0 \end{pmatrix}, \text{ with domain } \text{dom}(H_0) := \text{dom}(B_0^2) \oplus \text{dom}(B_0) = \mathbf{H}^2(\mathbb{R}^3) \oplus \mathbf{H}^1(\mathbb{R}^3). \quad (2.14)$$

The free Klein-Gordon equation (2.12) is equivalent to the system

$$i \frac{\partial}{\partial t} \psi = H_0 \psi, \quad \psi \in \mathcal{H}_0,$$

with $\psi_1 = \phi$, and $\psi_2 = \frac{\partial}{\partial t} \phi$.

We denote by $F_W : \mathcal{H}_0 \mapsto L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ the unitary operator:

$$F_W := \begin{pmatrix} B_0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.15)$$

One can easily verify that $F_W H_0 F_W^{-1} = B_0 \beta$, with $\beta := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. Set Q and Q^{-1} the unitary matrices that diagonalize β :

$$Q \beta Q^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.16)$$

where $Q := 2^{-1/2} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$. We finally define the unitary operator $U : \mathcal{H}_0 \mapsto L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, $U := Q F_W$. It follows that

$$\hat{H}_0 := U H_0 U^{-1} = B_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.17)$$

In this representation the free Klein-Gordon equation (2.12) is equivalent to the system,

$$i \frac{\partial}{\partial t} \psi = \hat{H}_0 \psi, \quad \psi \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3). \quad (2.18)$$

The appropriate position operator, that gives the position of the quantum particle, is multiplication by the variable x in the diagonal representation of the Klein-Gordon equation (2.18). See [8], [31] and [32] for the issue of the position operator.

2.5 Interacting Hamiltonian

The interacting Klein-Gordon equation for a particle in Λ is given by,

$$\left(i \frac{\partial}{\partial t} - A_0 \right)^2 \phi = \left((\mathbf{p} - A)^2 + m^2 \right) \phi, \quad (2.19)$$

where $\phi : \mathbb{R} \times \Lambda \mapsto \mathbb{C}$ is the wave function. As in the free case we formulate (2.19) as a 2 by 2 system that is first order in time.

We denote by $B(\underline{A})^2 := (\mathbf{p} - A)^2 + m^2 - A_0^2$. In Subsection 3.2 in [8] we prove that $B(\underline{A})^2$ has a realization as a selfadjoint operator in $L^2(\Lambda)$ and that $B(\underline{A})^2 \geq \varepsilon$ with ε as in (2.3). We designate by

$$\mathcal{H}(\underline{A}) := \text{dom}(B(\underline{A})) \oplus L^2(\Lambda) \quad (2.20)$$

the Hilbert space with inner product: $\langle \phi, \psi \rangle_{\mathcal{H}(\underline{A})} := \langle B(\underline{A}) \phi_1, B(\underline{A}) \psi_1 \rangle + \langle \phi_2, \psi_2 \rangle$, for $\phi = (\phi_1, \phi_2)$, $\psi = (\psi_1, \psi_2)$. The inner product of $\mathcal{H}(\underline{A})$ is the sesquilinear form associated to the classical field energy of the interacting Klein-Gordon equation (2.19).

The interacting Hamiltonian is the self-adjoint operator (see Subsection 3.2 in [8]), defined in $\mathcal{H}(\underline{A})$,

$$H(\underline{A}) := \begin{pmatrix} 0 & i \\ -i B(\underline{A})^2 & 2A_0 \end{pmatrix}, \quad (2.21)$$

with domain $\text{dom}(H(\underline{A})) := \text{dom}(B(\underline{A})^2) \oplus \text{dom}(B(\underline{A}))$.

The interacting Klein-Gordon equation (2.19) is equivalent to the system,

$$i \frac{\partial}{\partial t} \psi = H(\underline{A}) \psi, \quad \psi \in \mathcal{H}(\underline{A}),$$

with $\psi_1 = \phi$, and $\psi_2 = \frac{\partial}{\partial t} \phi$.

3 Wave and Scattering Operators

3.1 Wave Operators

The wave operators are defined as follows:

$$W_{\pm}(\underline{A}) := \text{s-} \lim_{t \rightarrow \pm\infty} e^{itH(\underline{A})} J e^{-itH_0}, \quad (3.1)$$

provided that the strong limits exist. Here,

$$J := \begin{pmatrix} B(\underline{A})^{-1} \chi_{\Lambda} B_0 & 0 \\ 0 & \chi_{\Lambda} \end{pmatrix}, \quad (3.2)$$

is a bounded identification operator from \mathcal{H}_0 into $\mathcal{H}(\underline{A})$.

3.1.1 Existence of the Wave Operators

In this section we prove existence of wave operators (3.1) for every magnetic potential $A \in \mathcal{A}_{\Phi}^{\text{LR}}(B)$. We provide additionally a change of gauge formula.

We first state a Lemma we that use, it is proved in Lemma 3.26 in [8].

LEMMA 3.1. *We denote by $\mathbf{v} : \mathbb{R}^3 \mapsto \mathbb{R}^3$,*

$$\mathbf{v}(p) := \frac{p}{(p^2 + m^2)^{1/2}}, \quad (3.3)$$

the function that associates to each momentum p the corresponding velocity. Take $x_0 \in \mathbb{R}^3 \setminus \{0\}$, $r_0 \in (0, \frac{1}{2}|x_0|)$ and $f \in \mathcal{S}(\mathbb{R}^3)$ be such that $\mathbf{v}(\text{supp}(f)) \subset B(x_0; r_0)$. For every $l \in \mathbb{N}$ there is a constant C_l such that

$$\left\| \chi_{(B(0; |t|r_0/2) + tB(x_0; r_0))} e^{-itB_0} f(\mathbf{p}) \chi_{B(0; |t|r_0/2)} \right\| \leq C_l (1 + |t|)^{-l}, \quad (3.4)$$

where χ_O is the characteristic function of the set O . Moreover, let $\tau \in C_0^\infty(\mathbb{R}^3; [0, 1])$ be such that $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$ and it vanishes for $|x| \geq 1$. There exists $v_0 > 0$ and a constant C_l , for every $l \in \mathbb{N}$, such that

$$\left\| \chi_{B(vt; |t|/2)} e^{-itB_0} \tau\left(\frac{16(\mathbf{p} - v\nu)}{v}\right) \chi_{B(0; |t|/8)} \right\| \leq C_l (1 + |t|)^{-l}, \quad (3.5)$$

for every $\nu \in \mathbb{S}^2$ and every $v \geq v_0$.

LEMMA 3.2. Let $A = \nabla \lambda \in \mathcal{A}_0^{\text{LR}}(0)$. Extend λ to a C^1 function in \mathbb{R}^3 , without changing notation. Set λ_∞ as in (2.10). Then,

$$s - \lim_{t \rightarrow \infty} e^{\pm i \lambda_\infty (\mathbf{p}/(\mathbf{p}^2 + m^2)^{1/2} + \mathbf{x}/t)} = e^{\pm i \lambda_\infty (\mathbf{p}/(\mathbf{p}^2 + m^2)^{1/2})} = e^{\pm i \lambda_\infty (\mathbf{p})}, \quad (3.6)$$

where the strong limit is taken in $L^2(\mathbb{R}^3)$. We recall that \mathbf{x} is the multiplication-by- \mathbf{x} operator.

Proof: We prove the assertion taking the minus sign, the proof with the plus sign is the same. The second equality is obvious because λ_∞ is homogeneous of degree 0. The spectral measure of the operator \mathbf{x}/t is the projection-valued measure $P_{\mathbf{x}/t}$ that associates to each Borel set \mathcal{B} $P_{\mathbf{x}/t}(\mathcal{B}) = \chi_{t\mathcal{B}}(\mathbf{x})$, $x \in \mathbb{R}^3$. As

$$\mathbf{p}/(\mathbf{p}^2 + m^2)^{1/2} + \mathbf{x}/t = e^{itB_0} \mathbf{x}/t e^{-itB_0}, \quad (3.7)$$

the corresponding spectral measure of this operator is given by $P_{\mathbf{p}/(\mathbf{p}^2 + m^2)^{1/2} + \mathbf{x}/t}(\mathcal{B}) = e^{itB_0} \chi_{t\mathcal{B}} e^{-itB_0}$, for every Borel set \mathcal{B} . Let $\phi \in \mathcal{S}(\mathbb{R})^3$ with $\hat{\phi}f = \hat{\phi}$, with f satisfying the hypotheses of in Lemma 3.1. Then, using (3.4) and the decay of ϕ , we prove that

$$\|P_{\mathbf{p}/(\mathbf{p}^2 + m^2)^{1/2} + \mathbf{x}/t}(B(0; r_0/4))\phi\| = \|\chi_{tB(0; r_0/4)} e^{-itB_0} f(\mathbf{p})(1 - \chi_{B(0; |t|r_0/2)} + \chi_{B(0; |t|r_0/2)})\phi\| \leq C \frac{1}{1 + |t|}. \quad (3.8)$$

Let $g_{r_0} : \mathbb{R}^3 \rightarrow \mathbb{C}$ be a continuous, bounded by 1, function that equals $e^{-i\lambda_\infty}$ in the complement of $B(0; r_0/4)$. Eq. (3.8) implies that

$$\lim_{t \rightarrow \infty} \left\| \left[e^{-i\lambda_\infty (\mathbf{p}/(\mathbf{p}^2 + m^2)^{1/2} + \mathbf{x}/t)} - g_{r_0}(\mathbf{p}/(\mathbf{p}^2 + m^2)^{1/2} + \mathbf{x}/t) \right] \phi \right\| = 0. \quad (3.9)$$

As g_{r_0} is continuous and $\mathbf{p}/(\mathbf{p}^2 + m^2)^{1/2} + \mathbf{x}/t \rightarrow \mathbf{p}/(\mathbf{p}^2 + m^2)^{1/2}$ in the strong resolvent sense, Theorem VIII.20 in [24] implies that

$$s - \lim_{t \rightarrow \infty} g_{r_0}(\mathbf{p}/(\mathbf{p}^2 + m^2)^{1/2} + \mathbf{x}/t) = g_{r_0}(\mathbf{p}/(\mathbf{p}^2 + m^2)^{1/2}). \quad (3.10)$$

Varying x_0 , r_0 and g_{r_0} and using (3.9)-(3.10) we obtain Eq. (3.6). \square

LEMMA 3.3. Set $A^{(1)}, A^{(2)} \in \mathcal{A}_\phi^{\text{LR}}(B)$. Take $\lambda \in \mathbb{C}^1(\mathbb{R}^3)$ such that $A^{(2)}(x) - A^{(1)}(x) = \nabla \lambda(x)$, $x \in \bar{\Lambda}$ [see (2.9)]. Then

$$s - \lim_{t \rightarrow \pm\infty} \left[B_0 \kappa'(\mathbf{x}) e^{-i\lambda(\mathbf{x})} B_0^{-1} - e^{-i\lambda_\infty(\pm\mathbf{p})} \right] e^{-itB_0} = 0, \quad s - \lim_{t \rightarrow \pm\infty} \left[\kappa'(\mathbf{x}) e^{-i\lambda(\mathbf{x})} - e^{-i\lambda_\infty(\pm\mathbf{p})} \right] e^{-itB_0} = 0 \quad (3.11)$$

here the strong limit is taken in $L^2(\mathbb{R}^3)$ and $\kappa' \in C^\infty(\mathbb{R}^3)$ is such that $1 - \kappa'$ is compactly supported. Recall the \mathbf{x} is the multiplication-by- \mathbf{x} operator.

Proof: We prove assertion using the plus sign. The proof with the minus sign is the same. We only prove the first equation in (3.11), which is the difficult part (proving the second uses the similar arguments). Take $\phi \in \mathcal{S}(\mathbb{R}^3)$ such that $f(p)\hat{\phi}(p) = \hat{\phi}(p)$, for some f satisfying the hypotheses of Lemma 3.1.

We have that

$$\begin{aligned}
\lim_{t \rightarrow \infty} \left\| B_0 \left[\kappa'(\mathbf{x}) e^{-i\lambda(\mathbf{x})} B_0^{-1} - B_0^{-1} e^{-i\lambda_\infty(\mathbf{p})} \right] e^{-itB_0} \phi \right\| &\leq C \lim_{t \rightarrow \infty} \left\| \left[\kappa'(\mathbf{x}) e^{-i\lambda(\mathbf{x})} B_0^{-1} - B_0^{-1} e^{-i\lambda_\infty(\mathbf{p})} \right] e^{-itB_0} \phi \right\|_{\mathbf{H}^1(\mathbb{R}^3)} \\
&\leq C \lim_{t \rightarrow \infty} \left\| [\mathbf{p}, \kappa'(\mathbf{x}) e^{-i\lambda(\mathbf{x})}] B_0^{-1} e^{-itB_0} \phi \right\| + C \lim_{t \rightarrow \infty} \left\| \left[\kappa'(\mathbf{x}) e^{-i\lambda(\mathbf{x})} - e^{-i\lambda_\infty(\mathbf{p})} \right] e^{-itB_0} \mathbf{p} B_0^{-1} \phi \right\| \\
&+ C \lim_{t \rightarrow \infty} \left\| \left[\kappa'(\mathbf{x}) e^{-i\lambda(\mathbf{x})} - e^{-i\lambda_\infty(\mathbf{p})} \right] e^{-itB_0} B_0^{-1} \phi \right\|.
\end{aligned} \tag{3.12}$$

As $\nabla \lambda(x) = A^{(2)}(x) - A^{(1)}(x)$ decays as $\frac{1}{|x|}$ as $|x|$ tends to infinity, the commutator $[\mathbf{p}, \kappa'(\mathbf{x}) e^{-i\lambda(\mathbf{x})}]$ decays as $\frac{1}{|x|}$ as $|x|$ tends to infinity, which together with Lemma 3.1 (or just the Rellich-Kondrakov lemma) imply that

$$s-\lim_{t \rightarrow \infty} [\mathbf{p}, \kappa'(\mathbf{x}) e^{-i\lambda(\mathbf{x})}] B_0^{-1} e^{-itB_0} \phi = 0. \tag{3.13}$$

Then, we obtain, using (3.12)-(3.13), that (3.11) is valid whenever

$$s-\lim_{t \rightarrow \infty} \left[\kappa'(\mathbf{x}) e^{-i\lambda(\mathbf{x})} - e^{-i\lambda_\infty(\mathbf{p})} \right] e^{-itB_0} \phi = s-\lim_{t \rightarrow \infty} e^{itB_0} \left[e^{-i\lambda(\mathbf{x})} - e^{-i\lambda_\infty(\mathbf{p})} \right] e^{-itB_0} \phi = 0, \tag{3.14}$$

in $L^2(\mathbb{R}^3)$, for every ϕ as above. We use Lemma 3.1 (or just the Rellich-Kondrakov lemma) to prove that

$$s-\lim_{t \rightarrow \infty} (1 - \kappa'(\mathbf{x})) e^{-i\lambda(\mathbf{x})} e^{-itB_0} \phi = 0, \tag{3.15}$$

which implies the first equality in (3.14). Additionally, Lemma 3.1 and the decay of ϕ imply that (here we use the notation of the referred lemma)

$$s-\lim_{t \rightarrow \infty} \left[e^{-i\lambda(\mathbf{x})} - e^{-i\lambda_\infty(\mathbf{x})} \right] e^{-itB_0} \phi = s-\lim_{t \rightarrow \infty} \chi_{(B(0;|t|r_0/2)+tB(x_0;r_0))} \left[e^{-i\lambda(\mathbf{x})} - e^{-i\lambda_\infty(\mathbf{x})} \right] e^{-itB_0} \phi. \tag{3.16}$$

Eq. (2.11) implies that there is a positive decreasing function $h : [0, \infty) \mapsto [0, \infty)$ with $\lim_{r \rightarrow \infty} h(r) = 0$, such that

$$\left\| \chi_{(B(0;|t|r_0/2)+tB(x_0;r_0))} \left[e^{-i\lambda(\mathbf{x})} - e^{-i\lambda_\infty(\mathbf{x})} \right] \right\| \leq h(t), \tag{3.17}$$

from which, together with (3.16), we get

$$s-\lim_{t \rightarrow \infty} e^{itB_0} \left[e^{-i\lambda(\mathbf{x})} - e^{-i\lambda_\infty(\mathbf{x})} \right] e^{-itB_0} \phi = 0. \tag{3.18}$$

Using (3.7) and the fact that λ_∞ is homogeneous of degree 0 we prove

$$s-\lim_{t \rightarrow \infty} e^{itB_0} e^{-i\lambda_\infty(\mathbf{x})} e^{-itB_0} \phi = s-\lim_{t \rightarrow \infty} e^{itB_0} e^{-i\lambda_\infty(\mathbf{x}/t)} e^{-itB_0} \phi = s-\lim_{t \rightarrow \infty} e^{-i\lambda_\infty(\mathbf{p}/(\mathbf{p}^2+m^2)^{1/2}+\mathbf{x}/t)} \phi = e^{-i\lambda_\infty(\mathbf{p})} \phi, \tag{3.19}$$

where we used Lemma 3.2. Eqs. (3.15), (3.18) and (3.19) imply (3.14), which in turn implies the desired result.

THEOREM 3.4 (Existence of Wave Operators and Change of Gauge Formula). *For every $\underline{A} = (A_0, A)$ with $A \in A_\phi^{\text{LR}}(B)$ the limits (3.1) exist and are isometric. For every $\underline{A}^{(i)} = (A_0, A^{(i)})$, $i \in \{1, 2\}$, with $A^{(1)}, A^{(2)} \in A_\phi^{\text{LR}}(B)$*

$$W_\pm(\underline{A}^{(2)}) = e^{i\lambda(\mathbf{x})} W_\pm(\underline{A}^{(1)}) U^{-1} \begin{pmatrix} e^{-i\lambda_\infty(\pm \mathbf{p})} & 0 \\ 0 & e^{-i\lambda_\infty(\mp \mathbf{p})} \end{pmatrix} U, \tag{3.20}$$

where $A^{(2)} - A^{(1)} = \nabla \lambda$ [see (2.9)]. Notice that in the expression above $e^{i\lambda(\mathbf{x})}$ is an operator from $\mathcal{H}(\underline{A}^{(1)})$ to $\mathcal{H}(\underline{A}^{(2)})$.

Proof: We suppose that $A^{(1)} \in \mathcal{A}_{\Phi}^{\text{SR}}(B)$ (recall that by Remark 2.5 the set $\mathcal{A}_{\Phi}^{\text{SR}}(B)$ is not empty) . Theorem 3.4 and the proof of Lemma 3.2 in [8] assure that the wave operators $W_{\pm}(\underline{A}^{(1)})$ exist, are isometric and that

$$W_{\pm}(\underline{A}^{(1)}) := s\text{-}\lim_{t \rightarrow \pm\infty} e^{itH(\underline{A}^{(1)})} \kappa e^{-itH_0},$$

for any κ as in (2.5) and the text above it. By proving the change of gauge formula we prove the existence of $W_{\pm}(\underline{A}^{(2)})$. See Lemma 3.2 of [8] whose proof applies also in this case. Clearly, the existence and the isometry of the wave operators, and the change of gauge formula in the case of general $A^{(1)}$ and $A^{(2)}$ follows from the same result in the case when one of the potentials is in $\mathcal{A}_{\Phi}^{\text{SR}}(B)$. We prove the assertion for W_+ . The proof for W_- is analogous. A simple computation gives

$$H(\underline{A}^{(2)}) = e^{i\lambda(\mathbf{x})} H(\underline{A}^{(1)}) e^{-i\lambda(\mathbf{x})}, \quad (3.21)$$

which implies that

$$W_+(\underline{A}^{(2)}) = e^{i\lambda(\mathbf{x})} s\text{-}\lim_{t \rightarrow \infty} e^{itH(\underline{A}^{(1)})} \kappa e^{-itH_0} e^{itH_0} \kappa' e^{-i\lambda(\mathbf{x})} e^{-itH_0}, \quad (3.22)$$

whenever the limit exists. Here κ' satisfies the properties of κ (i.e. it satisfies (2.5) and the text above it). Additionally, we suppose that $\kappa'\kappa = \kappa$. Using (2.17) we obtain that

$$\begin{aligned} e^{itH_0} \kappa' e^{-i\lambda(\mathbf{x})} e^{-itH_0} &= U^{-1} \begin{pmatrix} e^{itB_0} & 0 \\ 0 & e^{-itB_0} \end{pmatrix} Q \begin{pmatrix} B_0 \kappa' e^{-i\lambda(\mathbf{x})} B_0^{-1} & 0 \\ 0 & \kappa' e^{-i\lambda(\mathbf{x})} \end{pmatrix} Q^{-1} \begin{pmatrix} e^{-itB_0} & 0 \\ 0 & e^{itB_0} \end{pmatrix} U \\ &= \frac{1}{2} U^{-1} \begin{pmatrix} e^{itB_0} & 0 \\ 0 & e^{-itB_0} \end{pmatrix} \begin{pmatrix} B_0 \kappa' e^{-i\lambda(\mathbf{x})} B_0^{-1} + \kappa' e^{-i\lambda(\mathbf{x})} & B_0 \kappa' e^{-i\lambda(\mathbf{x})} B_0^{-1} - \kappa' e^{-i\lambda(\mathbf{x})} \\ B_0 \kappa' e^{-i\lambda(\mathbf{x})} B_0^{-1} - \kappa' e^{-i\lambda(\mathbf{x})} & B_0 \kappa' e^{-i\lambda(\mathbf{x})} B_0^{-1} + \kappa' e^{-i\lambda(\mathbf{x})} \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} e^{-itB_0} & 0 \\ 0 & e^{itB_0} \end{pmatrix} U, \end{aligned} \quad (3.23)$$

which together with Lemma 3.3 imply that

$$s\text{-}\lim_{t \rightarrow \infty} e^{itH_0} \kappa' e^{-i\lambda(\mathbf{x})} e^{-itH_0} = U^{-1} \begin{pmatrix} e^{-i\lambda_{\infty}(\mathbf{p})} & 0 \\ 0 & e^{-i\lambda_{\infty}(-\mathbf{p})} \end{pmatrix} U, \quad (3.24)$$

from which the desired result follows. \square

3.2 Scattering Operator

The scattering operator is defined, for every $\underline{A} = (A_0, A)$ (with $A \in \mathcal{A}_{\Phi}^{\text{LR}}(B)$) by

$$S(\underline{A}) = W_+^*(\underline{A}) W_-(\underline{A}). \quad (3.25)$$

The following theorem gives the change of gauge formula for the scattering operator.

THEOREM 3.5. *For every $\underline{A}^{(i)} = (A_0, A^{(i)})$, $i \in \{1, 2\}$, with $A^{(1)}, A^{(2)} \in \mathcal{A}_{\Phi}^{\text{LR}}(B)$,*

$$S(\underline{A}^{(2)}) = U^{-1} \begin{pmatrix} e^{i\lambda_{\infty}(\mathbf{p})} & 0 \\ 0 & e^{i\lambda_{\infty}(-\mathbf{p})} \end{pmatrix} U S(\underline{A}^{(1)}) U^{-1} \begin{pmatrix} e^{-i\lambda_{\infty}(-\mathbf{p})} & 0 \\ 0 & e^{-i\lambda_{\infty}(\mathbf{p})} \end{pmatrix} U, \quad (3.26)$$

where $A^{(2)} - A^{(1)} = \nabla \lambda$ [see (2.9)].

Proof: The result is a direct consequence of Theorem 3.4. Notice that the dual of the operator $e^{i\lambda(\mathbf{x})} : \mathcal{H}(\underline{A}^{(1)}) \mapsto \mathcal{H}(\underline{A}^{(2)})$ is $e^{-i\lambda(\mathbf{x})} : \mathcal{H}(\underline{A}^{(2)}) \mapsto \mathcal{H}(\underline{A}^{(1)})$. \square

In [4] we considered change of gauge formulae where the fluxes can differ in multiples of 2π , in the case of the Schrödinger equation. Similar results are true for the Klein-Gordon equation.

4 High Momenta Limit of the Scattering Operator

In this section we prove one of our main results: We give a high-momenta expression for the scattering operator, with error bounds. This formula is the content of Theorem 4.1, which is a generalization of Theorem 2.8 in [8] to long-range magnetic potentials. Our formula is used to reconstruct important information from the potentials and the magnetic field.

For every $\nu \in \mathbb{S}^1$ we denote by (see Eq. (2.5))

$$\Lambda_\nu := \left\{ x \in \Lambda : x + \tau\nu \in \Lambda, \forall \tau \in \mathbb{R} \right\}, \text{ and } \Lambda_{\kappa, \nu} := \left\{ x \in \Lambda_\nu : \kappa(x + \tau\nu) = 1, \forall \tau \in \mathbb{R} \right\}. \quad (4.1)$$

THEOREM 4.1. *Set $\nu \in \mathbb{S}^2$ and $l \in \mathbb{N}$, $l \geq \zeta/2, l \geq 2$. Suppose that $\phi, \psi \in \mathbf{H}_{\langle x \rangle^{4l}}^2(\mathbb{R}^3)^2$ are supported in $\Lambda_{\kappa, \nu}$. Let $\underline{A} = (A_0, A)$, with $A \in A_\Phi^{\text{LR}}(B)$. Suppose that for some $\zeta_0 > 1$ there is $A^{(1)} \in A_\Phi^{\text{SR}}(B)$ satisfying (2.7) with $\zeta = \zeta_0$. Then*

$$\begin{aligned} \langle US(\underline{A})U^{-1}e^{i\mathbf{x} \cdot \nu\nu}\phi, e^{i\mathbf{x} \cdot \nu\nu}\psi \rangle_{L^2(\mathbb{R}^3)^2} = & \left\langle \begin{pmatrix} e^{i \int_{-\infty}^{\infty} dr(A \cdot \nu - A_0)(\mathbf{x} + r\nu)} & 0 \\ 0 & e^{-i \int_{-\infty}^{\infty} dr(A \cdot \nu + A_0)(\mathbf{x} + r\nu)} \end{pmatrix} \phi, \psi \right\rangle, \\ & + \|\phi\|_{\mathbf{H}_{\langle x \rangle^{4l}}^2(\mathbb{R}^3)} \|\psi\|_{\mathbf{H}_{\langle x \rangle^{4l}}^2(\mathbb{R}^3)} \begin{cases} O\left(v^{1-\zeta_0} + \frac{1}{v}\right), & \text{if } \zeta_0 \neq 2, \\ O\left(\frac{\ln(v)}{v}\right), & \text{if } \zeta_0 = 2. \end{cases} \end{aligned} \quad (4.2)$$

Recall that by Remark 2.5 the set $A_\Phi^{\text{SR}}(B)$ is not empty, i.e. there is always a potential (the Coulomb potential) in $A_\Phi^{\text{SR}}(B)$ that satisfies (2.7) for some $\zeta > 1$ that depends on the decay rate, μ , of the magnetic field. See (2.2).

Proof of the Theorem: We identify $A^{(2)} \equiv A \in \mathcal{A}_\Phi^{\text{LR}}(B)$. Set λ such that $A^{(2)} - A^{(1)} = \nabla\lambda$ and take $\underline{A}^{(j)} = (A_0, A^{(j)})$, $j \in \{1, 2\}$. By the fact that λ_∞ is homogeneous of degree 0 and Lemma 3.8 in [4]

$$|\lambda_\infty(x + y) - \lambda_\infty(x)| \leq C|y| \quad (4.3)$$

for every $x \in \mathbb{R}^3 \setminus \{0\}$ and every y with $|y| \leq \frac{|x|}{2}$. Take $\tau \in C_0^\infty(\mathbb{R}^3; [0, 1])$ be such that $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$ and it vanishes for $|x| \geq 1$. We have that, for every $\varphi \in \mathbf{H}^1(\mathbb{R}^3)$,

$$\begin{aligned} \left\| [e^{-i\mathbf{x} \cdot \nu\nu} e^{i\lambda_\infty(\mathbf{p})} e^{i\mathbf{x} \cdot \nu\nu} - e^{i\lambda_\infty(\nu)}] \varphi \right\| & \leq \left\| [e^{i\lambda_\infty(\mathbf{p}/v + \nu)} - e^{i\lambda_\infty(\nu)}] \tau(2\mathbf{p}/v) \varphi \right\| + 2 \left\| (1 - \tau(2\mathbf{p}/v)) \frac{1}{\langle \mathbf{p} \rangle} \right\| \|\varphi\|_{\mathbf{H}^1(\mathbb{R}^3)} \\ & \leq C \frac{1}{v} \|\varphi\|_{\mathbf{H}^1(\mathbb{R}^3)}, \end{aligned} \quad (4.4)$$

where we use (4.3). By Theorem 3.5, (4.4) and arguments similar to the ones used in the proof of (4.4), we obtain

$$\begin{aligned}
& \langle e^{-i\mathbf{x}\cdot v\nu} US(\underline{A}^{(2)})U^{-1}e^{i\mathbf{x}\cdot v\nu}\phi, \psi \rangle \\
&= \left\langle e^{-i\mathbf{x}\cdot v\nu} \begin{pmatrix} e^{i\lambda_\infty(\mathbf{p})} & 0 \\ 0 & e^{i\lambda_\infty(-\mathbf{p})} \end{pmatrix} US(\underline{A}^{(1)})U^{-1} \begin{pmatrix} e^{-i\lambda_\infty(-\mathbf{p})} & 0 \\ 0 & e^{-i\lambda_\infty(\mathbf{p})} \end{pmatrix} e^{i\mathbf{x}\cdot v\nu}\phi, \psi \right\rangle \\
&= \left\langle \begin{pmatrix} e^{i\lambda_\infty(\nu)} & 0 \\ 0 & e^{i\lambda_\infty(-\nu)} \end{pmatrix} e^{-i\mathbf{x}\cdot v\nu} US(\underline{A}^{(1)})U^{-1}e^{i\mathbf{x}\cdot v\nu} \begin{pmatrix} e^{-i\lambda_\infty(-\nu)} & 0 \\ 0 & e^{-i\lambda_\infty(\nu)} \end{pmatrix} \phi, \psi \right\rangle \\
&+ \|\phi\|_{\mathbf{H}^1(\mathbb{R}^3)^2} \|\psi\|_{\mathbf{H}^1(\mathbb{R}^3)^2} O\left(\frac{1}{v}\right).
\end{aligned} \tag{4.5}$$

Using Theorem 2.8 in [8] we obtain

$$\begin{aligned}
& \left\langle \begin{pmatrix} e^{i\lambda_\infty(\nu)} & 0 \\ 0 & e^{i\lambda_\infty(-\nu)} \end{pmatrix} e^{-i\mathbf{x}\cdot v\nu} US(\underline{A}^{(1)})U^{-1}e^{i\mathbf{x}\cdot v\nu} \begin{pmatrix} e^{-i\lambda_\infty(-\nu)} & 0 \\ 0 & e^{-i\lambda_\infty(\nu)} \end{pmatrix} \phi, \psi \right\rangle_{L^2(\mathbb{R}^3)^2} \\
&= \left\langle \begin{pmatrix} e^{i\lambda_\infty(\nu)} & 0 \\ 0 & e^{i\lambda_\infty(-\nu)} \end{pmatrix} \begin{pmatrix} e^{i\int_{-\infty}^{\infty} dr(A^{(1)}\cdot\nu-A_0)(\mathbf{x}+r\nu)} & 0 \\ 0 & e^{-i\int_{-\infty}^{\infty} dr(A^{(1)}\cdot\nu+A_0)(\mathbf{x}+r\nu)} \end{pmatrix} \begin{pmatrix} e^{-i\lambda_\infty(-\nu)} & 0 \\ 0 & e^{-i\lambda_\infty(\nu)} \end{pmatrix} \phi, \psi \right\rangle_{L^2(\mathbb{R}^3)^2} \\
&+ \|\phi\|_{\mathbf{H}^2_{\langle\mathbf{x}\rangle^{41}}(\mathbb{R}^3)^2} \|\psi\|_{\mathbf{H}^2_{\langle\mathbf{x}\rangle^{41}}(\mathbb{R}^3)^2} \begin{cases} O\left(v^{1-\zeta_0} + \frac{1}{v}\right), & \text{if } \zeta_0 \neq 2, \\ O\left(\frac{\ln(v)}{v}\right), & \text{if } \zeta_0 = 2. \end{cases}
\end{aligned} \tag{4.6}$$

We get the desired result using Eqs. (4.5)-(4.6) and

$$\begin{aligned}
& \begin{pmatrix} e^{i\lambda_\infty(\nu)} & 0 \\ 0 & e^{i\lambda_\infty(-\nu)} \end{pmatrix} \begin{pmatrix} e^{i\int_{-\infty}^{\infty} dr(A^{(1)}\cdot\nu-A_0)(\mathbf{x}+r\nu)} & 0 \\ 0 & e^{-i\int_{-\infty}^{\infty} dr(A^{(1)}\cdot\nu+A_0)(\mathbf{x}+r\nu)} \end{pmatrix} \begin{pmatrix} e^{-i\lambda_\infty(-\nu)} & 0 \\ 0 & e^{-i\lambda_\infty(\nu)} \end{pmatrix} \\
&= \begin{pmatrix} e^{i\int_{-\infty}^{\infty} dr(A^{(2)}\cdot\nu-A_0)(\mathbf{x}+r\nu)} & 0 \\ 0 & e^{-i\int_{-\infty}^{\infty} dr(A^{(2)}\cdot\nu+A_0)(\mathbf{x}+r\nu)} \end{pmatrix}.
\end{aligned} \tag{4.7}$$

5 Reconstruction Methods: The magnetic Field, the Electric Potential and Magnetic Fluxes Modulo 2π

In this section we use the high momenta limit of the scattering operator in Theorem 4.1 to reconstruct the electric potential, the magnetic field and certain fluxes of the magnetic potential around handles of the obstacle. These results extend the results in [8] (Theorems 2.10 and 2.12), where they are proved only for short-range magnetic potentials. The main new ingredient that allows us to extend the results to the long-range case is Theorem 4.1. Once it is established, the proofs follows the same lines for both cases. Since they are already presented in [8], for the short-range case, we only state the results and refer to [8] for the proofs.

DEFINITION 5.1. We denote by Λ_{Rec} the set of points $x \in \Lambda$ such that, for some two-dimensional plane P_x , $x + P_x \subset (\kappa^{-1}(\{1\}))^o$, for some function $\kappa \in C_\infty(\mathbb{R}^3)$ satisfying (2.5) and the text above it.

THEOREM 5.2. *The high-momenta limit (4.2) of the scattering operator uniquely determines $B(y)$ and $A_0(y)$ for every $y \in \Lambda_{\text{Rec}}$.*

Proof: The proof follows the lines of the proof of Theorem 2.10 in [8], using Theorem 4.1. Note that the proof also gives a method for the unique reconstruction of $B(y)$ and $A_0(y)$ for every $y \in \Lambda_{\text{Rec}}$. \square

We denote by

$$L(x, \hat{\mathbf{v}}) := x + \mathbb{R}\hat{\mathbf{v}}, \quad (5.1)$$

for every $x \in \mathbb{R}^3$ and any unit vector $\hat{\mathbf{v}} \in \mathbb{S}^2$. Suppose that $L(x, \hat{\mathbf{v}}) \cup L(y, \hat{\mathbf{w}}) \subset \Lambda$, $x \cdot y \geq 0$, and $\rho > 0$ is such that

$$\text{convex}((x + (-\infty, -\rho]\hat{\mathbf{v}}) \cup (y + (-\infty, -\rho]\hat{\mathbf{w}})) \cup \text{convex}((x + [\rho, \infty)\hat{\mathbf{v}}) \cup (y + [\rho, \infty)\hat{\mathbf{w}})) \subset \mathbb{R}^3 \setminus B(0; r),$$

where $K \subset B(0; r)$ ($\text{convex}(\cdot)$ denotes the convex hull). We denote by $\gamma(x, y, \hat{\mathbf{v}}, \hat{\mathbf{w}})$ the curve with sides $x + [-\rho, \rho]\hat{\mathbf{v}}$, oriented in the direction of $\hat{\mathbf{v}}$, $y + [-\rho, \rho]\hat{\mathbf{w}}$, oriented in the direction of $-\hat{\mathbf{w}}$, and the straight lines that join the points $x + \rho\hat{\mathbf{v}}$ with $y + \rho\hat{\mathbf{w}}$ and $y - \rho\hat{\mathbf{w}}$ and $x - \rho\hat{\mathbf{v}}$. \square

THEOREM 5.3. *Suppose that $B = 0$ and that $A_0 = 0$. Then, for any flux, Φ , and all $A \in \mathcal{A}_\Phi^{LR}(0)$, the high-momenta limit of $S(\underline{A})$ in (4.2), known for $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$, determines the fluxes*

$$\int_{\gamma(x, y, \hat{\mathbf{v}}, \hat{\mathbf{w}})} A \quad (5.2)$$

modulo 2π , for all curves $\gamma(x, y, \hat{\mathbf{v}}, \hat{\mathbf{w}})$.

Proof: The proof follows the lines of the proof of Theorem 2.12 in [8], using Theorem 4.1. \square

6 Reconstruction Methods: Long-Range Magnetic Potentials

In this section we derive information of the long-range part of the magnetic potential from the high momenta limit of the scattering operator. We first introduce some definitions.

Take $R > 0$ such that $K \subset B(0; R)$. Suppose that $L(x, \hat{\mathbf{v}}) \subset \Lambda$, and $L(x, \hat{\mathbf{v}}) \cap B(0; R) \neq \emptyset$. We denote by $c(x, \hat{\mathbf{v}}) \equiv c_{R, \mathfrak{s}}(x, \hat{\mathbf{v}})$ the curve consisting of the segment $L(x, \hat{\mathbf{v}}) \cap \overline{B(0; R)}$ and a C^∞ simple differentiable curve \mathfrak{s} on $\partial\overline{B(0; R)}$ that connects the points $L(x, \hat{\mathbf{v}}) \cap \partial\overline{B(0; R)}$. We orient $c(x, \hat{\mathbf{v}})$ in such a way that the segment of straight line has the orientation of $\hat{\mathbf{v}}$. See Figure 2. We stress that $c_{R, \mathfrak{s}}(x, \hat{\mathbf{v}})$ depends on the curve we choose joining the points in $L(x, \hat{\mathbf{v}}) \cap \overline{B(0; R)}$. However, the quantities that we associate to it below do not depend on this election.

DEFINITION 6.1. For every magnetic potential $A \in \mathcal{A}_\Phi^{LR}(B)$ and every $\hat{\mathbf{v}} \in \mathbb{S}^2$ we define the quantity $\Phi_L(A, \hat{\mathbf{v}})$, which we name the long-range flux of A in the direction $\hat{\mathbf{v}}$, as follows: Suppose that $K \subset B(0; R)$. Take $x \in \mathbb{R}^3$ with $|x| \geq R$ and define the curve $c_{|x|+1, \mathfrak{s}}(x, \hat{\mathbf{v}})$ as above. Then, denoting $\mathfrak{s}(s) := s(\mathfrak{s} - x) + x$,

$$\Phi_L(A, \hat{\mathbf{v}}) := - \lim_{s \rightarrow \infty} \int_{\mathfrak{s}(s)} A. \quad (6.1)$$

We compute the limit in the right hand side of (6.1). Let us take $A^{(\text{SR})} \in \mathcal{A}_\Phi^{\text{SR}}(B)$ and λ such that $A - A^{(\text{SR})} = \nabla\lambda$. The fact that $A^{(\text{SR})}$ is short-range implies that

$$\lim_{s \rightarrow \infty} \int_{\mathfrak{s}(s)} A^{(\text{SR})} = 0, \quad (6.2)$$

and, therefore,

$$\lim_{s \rightarrow \infty} \int_{s(C(|x|+1), \mathfrak{s}(x, \hat{\mathbf{v}}) - x) + x} A - A^{(\text{SR})} = 0 = \int_{L(x, \hat{\mathbf{v}})} (A - A^{(\text{SR})}) + \lim_{s \rightarrow \infty} \int_{\mathfrak{s}(s)} (A - A^{(\text{SR})}) = \lambda_\infty(\hat{\mathbf{v}}) - \lambda_\infty(-\hat{\mathbf{v}}) + \lim_{s \rightarrow \infty} \int_{\mathfrak{s}(s)} A.$$

We conclude that

$$\Phi_L(A, \hat{\mathbf{v}}) = \lambda_\infty(\hat{\mathbf{v}}) - \lambda_\infty(-\hat{\mathbf{v}}). \quad (6.3)$$

From the definition in Eq. (6.1) it follows that $\Phi_L(A, \hat{\mathbf{v}})$ is an intrinsic property of A , although it can also be expressed as $\lambda_\infty(\hat{\mathbf{v}}) - \lambda_\infty(-\hat{\mathbf{v}})$, as (6.3) shows. It is also clear from (6.3) that $\Phi_L(A, \hat{\mathbf{v}})$ does not depend on the particular curve \mathfrak{s} that is used to define it.

PROPOSITION 6.2. *Suppose that $B \in C^2(\bar{\Lambda})$ and that $|B(x)| \leq C \frac{1}{(1+|x|)^\mu}$, $|\frac{\partial}{\partial x_i} B(x)| \leq C \frac{1}{(1+|x|)^{\mu+1}}$, $|\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} B(x)| \leq C \frac{1}{(1+|x|)^{\mu+2}}$, for every $i, j \in \{1, 2, 3\}$ and every $x \in \Lambda$. Let $\delta > 1$ and $A \in \mathcal{A}_{\Phi, \delta}(B)$. Then,*

$$A_\infty(\hat{\mathbf{v}}) := \lim_{\tau \rightarrow \infty} A(\tau \hat{\mathbf{v}}) \tau \quad (6.4)$$

exists, it is continuous as a function of $\hat{\mathbf{v}} \in \mathbb{S}^2$ and $\forall \hat{\mathbf{v}} \in \mathbb{S}^2 : A_\infty(\hat{\mathbf{v}}) \cdot \hat{\mathbf{v}} = 0$. We extend (6.4) to $\mathbb{R}^3 \setminus \{0\}$ taking

$$A_\infty(x) := \frac{1}{|x|} A_\infty\left(\frac{x}{|x|}\right). \quad (6.5)$$

Proof: The proof follows from Corollary 3.13 in [7]. Although in [7] only 2 dimensions are considered, the proof also applies in our case.

PROPOSITION 6.3. *Suppose that $B \in C^2(\bar{\Lambda})$ and that $|B(x)| \leq C \frac{1}{(1+|x|)^\mu}$, $|\frac{\partial}{\partial x_i} B(x)| \leq C \frac{1}{(1+|x|)^{\mu+1}}$, $|\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} B(x)| \leq C \frac{1}{(1+|x|)^{\mu+2}}$, for every $i, j \in \{1, 2, 3\}$ and every $x \in \Lambda$. Let $\delta > 1$ and $A \in \mathcal{A}_{\Phi, \delta}(B)$, then for every $\hat{\mathbf{v}} \in \mathbb{S}^2$*

$$\Phi_L(A, \hat{\mathbf{v}}) = - \int_0^\pi A_\infty\left(\cos(\theta)\hat{\mathbf{v}} + \sin(\theta)\hat{\mathbf{v}}^\perp\right) \cdot \left(-\sin(\theta)\hat{\mathbf{v}} + \cos(\theta)\hat{\mathbf{v}}^\perp\right) d\theta, \quad (6.6)$$

$$\frac{\partial}{\partial \vartheta} \Phi_L\left(A, \cos(\vartheta)\hat{\mathbf{v}} + \sin(\vartheta)\hat{\mathbf{v}}^\perp\right) \Big|_{\vartheta=0} = \left[A_\infty(\hat{\mathbf{v}}) + A_\infty(-\hat{\mathbf{v}})\right] \cdot \hat{\mathbf{v}}^\perp, \quad (6.7)$$

where $\hat{\mathbf{v}}^\perp$ is any unit vector, orthogonal to $\hat{\mathbf{v}}$.

Proof: Notice that, conveniently selecting \mathfrak{s} ,

$$\lim_{s \rightarrow \infty} \int_{\mathfrak{s}(s)} A = \lim_{s \rightarrow \infty} \int_0^\pi A\left(s \cos(\theta)\hat{\mathbf{v}} + s \sin(\theta)\hat{\mathbf{v}}^\perp\right) \cdot \left(-s \sin(\theta)\hat{\mathbf{v}} + s \cos(\theta)\hat{\mathbf{v}}^\perp\right) d\theta \quad (6.8)$$

and that, by (2.6), $A\left(s \cos(\theta)\hat{\mathbf{v}} + s \sin(\theta)\hat{\mathbf{v}}^\perp\right) \cdot \left(s \cos(\theta)\hat{\mathbf{v}} + s \sin(\theta)\hat{\mathbf{v}}^\perp\right)$ is uniformly bounded (with respect to s). Eq. (6.6) follows from the Lebesgue convergence theorem, (6.1) and (6.4). Deriving (6.6) we obtain Eq. (6.7).

LEMMA 6.4. *Suppose that $B \in C^2(\bar{\Lambda})$ and that $|B(x)| \leq C \frac{1}{(1+|x|)^\mu}$, $|\frac{\partial}{\partial x_i} B(x)| \leq C \frac{1}{(1+|x|)^{\mu+1}}$, $|\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} B(x)| \leq C \frac{1}{(1+|x|)^{\mu+2}}$, for every $i, j \in \{1, 2, 3\}$ and every $x \in \Lambda$. Let $\delta > 1$ and $A \in \mathcal{A}_{\Phi, \delta}(B)$, then for $\hat{\mathbf{v}} \in \mathbb{S}^2$ the function*

$$\vartheta \mapsto \int_{L(x, \cos(\vartheta)\hat{\mathbf{v}} + \sin(\vartheta)\hat{\mathbf{v}}^\perp)} A,$$

for $x \in \Lambda_{\hat{\mathbf{v}}}$, is differentiable and

$$\frac{\partial}{\partial \vartheta} \int_{L(x, \cos(\vartheta)\hat{\mathbf{v}} + \sin(\vartheta)\hat{\mathbf{v}}^\perp)} A \Big|_{\vartheta=0} = \int_{-\infty}^{\infty} \tau B(x + \tau \hat{\mathbf{v}}) \cdot (\hat{\mathbf{v}}^\perp \times \hat{\mathbf{v}}) d\tau + [A_\infty(\hat{\mathbf{v}}) + A_\infty(-\hat{\mathbf{v}})] \cdot \hat{\mathbf{v}}^\perp. \quad (6.9)$$

Proof: Take $A^{(\text{SR})} \in \mathcal{A}_\Phi^{\text{SR}}(B)$ and λ such that $A - A^{(\text{SR})} = \nabla \lambda$. Then we have

$$\begin{aligned} \int_{L(x, \cos(\vartheta)\hat{\mathbf{v}} + \sin(\vartheta)\hat{\mathbf{v}}^\perp)} A &= \lambda_\infty(\cos(\vartheta)\hat{\mathbf{v}} + \sin(\vartheta)\hat{\mathbf{v}}^\perp) - \lambda_\infty(-(\cos(\vartheta)\hat{\mathbf{v}} + \sin(\vartheta)\hat{\mathbf{v}}^\perp)) + \int_{L(x, \cos(\vartheta)\hat{\mathbf{v}} + \sin(\vartheta)\hat{\mathbf{v}}^\perp)} A^{(\text{SR})} \\ &= \Phi_L(A, \cos(\vartheta)\hat{\mathbf{v}} + \sin(\vartheta)\hat{\mathbf{v}}^\perp) + \int_{L(x, \cos(\vartheta)\hat{\mathbf{v}} + \sin(\vartheta)\hat{\mathbf{v}}^\perp)} A^{(\text{SR})}. \end{aligned} \quad (6.10)$$

We conclude using Proposition 6.3 and the fact that

$$\frac{\partial}{\partial \vartheta} \int_{L(x, \cos(\vartheta)\hat{\mathbf{v}} + \sin(\vartheta)\hat{\mathbf{v}}^\perp)} A^{(\text{SR})} \Big|_{\vartheta=0} = \int_{-\infty}^{\infty} \tau B(x + \tau \hat{\mathbf{v}}) \cdot (\hat{\mathbf{v}}^\perp \times \hat{\mathbf{v}}) d\tau, \quad (6.11)$$

which is a direct consequence of Stokes' theorem. \square

THEOREM 6.5. *Suppose that $\underline{A} = (V, A)$, $A \in \mathcal{A}_\Phi^{\text{LR}}(B)$. The high-momenta limit (4.2) of the scattering operator $S(\underline{A})$ uniquely determines $\Phi_L(A, \hat{\mathbf{v}})$ modulo π , for every $\hat{\mathbf{v}} \in \mathbb{S}^2$ (in the case $A_0 = 0$, it uniquely determines $\Phi_L(A, \hat{\mathbf{v}})$ modulo 2π). If we, furthermore, assume that $A \in \mathcal{A}_{\Phi, \delta}(B)$ (for some $\delta > 1$), and that $B \in C^2(\bar{\Lambda})$ is such that $|B(x)| \leq C \frac{1}{(1+|x|)^\mu}$, $|\frac{\partial}{\partial x_i} B(x)| \leq C \frac{1}{(1+|x|)^{\mu+1}}$, $|\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} B(x)| \leq C \frac{1}{(1+|x|)^{\mu+2}}$, for every $i, j \in \{1, 2, 3\}$ and every $x \in \Lambda$, then high-momenta limit (4.2) of the scattering operator uniquely determines $A_\infty(\hat{\mathbf{v}}) + A_\infty(-\hat{\mathbf{v}})$, for all $\hat{\mathbf{v}} \in \mathbb{S}^2$.*

Proof: Take $x = R\hat{\mathbf{v}}^\perp$, where $(\kappa^{-1}(\{1\}))^c \subset B(0; R)$. From the high-momenta limit (4.2) of the scattering operator we uniquely determine $e^{-i \int_{-\infty}^{\infty} 2A \cdot \nu(x + \tau \hat{\mathbf{v}}) d\tau}$ (if $A_0 = 0$, we uniquely determine $e^{-i \int_{-\infty}^{\infty} A \cdot \nu(x + \tau \hat{\mathbf{v}}) d\tau}$). We recall the notation and procedures used in Definition 6.1. The result concerning $\Phi_L(A, \hat{\mathbf{v}})$ follows from the fact that

$$\int_{-\infty}^{\infty} A \cdot \nu(x + \tau \hat{\mathbf{v}}) d\tau = \Phi_L(A; \hat{\mathbf{v}}) + \int_{-\infty}^{\infty} A^{(\text{SR})} \cdot \nu(x + \tau \hat{\mathbf{v}}) d\tau = \Phi_L(A; \hat{\mathbf{v}}) + \lim_{s \rightarrow \infty} \int_{\mathcal{P}(x, \hat{\mathbf{v}}, s)} B, \quad (6.12)$$

where $\mathcal{P}(x, \hat{\mathbf{v}}, s)$ is a piece of flat plane whose boundary is $s(c_{(|x|+1), s}(x, \hat{\mathbf{v}}) - x) + x$. As B can be recovered in $\mathcal{P}(x, \hat{\mathbf{v}}, s)$ from the scattering operator, see Theorem 5.2, then the assertion follows. Now we prove part of the theorem that concerns $A_\infty(\hat{\mathbf{v}}) + A_\infty(-\hat{\mathbf{v}})$. It actually follows from Lemma 6.4, since

$$\frac{\partial}{\partial \vartheta} e^{-i \int_{L(x, \cos(\vartheta)\hat{\mathbf{v}} + \sin(\vartheta)\hat{\mathbf{v}}^\perp)} 2A} \Big|_{\vartheta=0} = 2ie^{-i \int_{-\infty}^{\infty} 2A \cdot \nu(x + \tau \hat{\mathbf{v}}) d\tau} \left[\int_{-\infty}^{\infty} \tau B(x + \tau \hat{\mathbf{v}}) \cdot (\hat{\mathbf{v}}^\perp \times \hat{\mathbf{v}}) d\tau + [A_\infty(\hat{\mathbf{v}}) + A_\infty(-\hat{\mathbf{v}})] \cdot \hat{\mathbf{v}}^\perp \right], \quad (6.13)$$

and we proved above that $e^{-i \int_{-\infty}^{\infty} 2A \cdot \nu(x + \tau \hat{\mathbf{v}}) d\tau}$ and $\int_{-\infty}^{\infty} \tau B(x + \tau \hat{\mathbf{v}}) \cdot (\hat{\mathbf{v}}^\perp \times \hat{\mathbf{v}}) d\tau$ can be recovered from the high-momenta limit of the scattering operator. Then we recover $[A_\infty(\hat{\mathbf{v}}) + A_\infty(-\hat{\mathbf{v}})] \cdot \hat{\mathbf{v}}^\perp$ for every $\hat{\mathbf{v}}^\perp$, orthonormal to $\hat{\mathbf{v}}$. As $A_\infty(\hat{\mathbf{v}}) \cdot \hat{\mathbf{v}} = 0$, for every $\hat{\mathbf{v}} \in \mathbb{S}^2$ (see Proposition 6.2), we recover $A_\infty(\hat{\mathbf{v}}) + A_\infty(-\hat{\mathbf{v}})$. \square

We now present some definitions and notation, first introduced in [4], see also [8] (we actually use a slight different notation). We do not give all details and motivations of our formalism, see Definitions 7.4, 7.5, 7.9 and 7.10 in [4] for a full detailed version: Take $R > 0$ such that $K \subset B(0; R)$. We define the following equivalence relation on $\Lambda_{\hat{\mathbf{v}}}$: We say

that $xR_{\hat{\mathbf{v}}}y$ if, and only if, $[c_{\|x\|+\|y\|+R,\mathfrak{s}}(x,\hat{\mathbf{v}})]_{H_1(\Lambda;\mathbb{R})} = [c_{\|x\|+\|y\|+R,\mathfrak{s}'}(y,\hat{\mathbf{v}})]_{H_1(\Lambda;\mathbb{R})}$ for every curves $\mathfrak{s}, \mathfrak{s}'$, here $H_1(\Lambda;\mathbb{R})$ is the one-singular-homology group in Λ with coefficients in \mathbb{R} . Notice that once the equality follows for some curves $\mathfrak{s}, \mathfrak{s}'$, it holds true for every such curves, because the sphere \mathbb{S}^2 is simply connected. We denote by $\{\Lambda_{\hat{\mathbf{v}},h}\}_{h \in \mathcal{I}}$ the partition of $\Lambda_{\hat{\mathbf{v}}}$ given by this equivalence relation (notice that it is an open disjoint cover of $\Lambda_{\hat{\mathbf{v}}}$). Observe that this equivalence relation, and the associated partition of $\Lambda_{\hat{\mathbf{v}}}$ coincide with the one given in [4], [8].

DEFINITION 6.6. Let $\Phi, A \in \mathcal{A}_{\Phi}^{\text{LR}}(0)$, $\hat{\mathbf{v}} \in \mathbb{S}^2$, and $h \in \mathcal{I}$. Take $R > 0$ such that $K \subset B(0; R)$. We define,

$$F_h := \int_{c_{\|x\|+R,\mathfrak{s}}(x,\hat{\mathbf{v}})} A,$$

where x is any point in $\Lambda_{\hat{\mathbf{v}},h}$. Note that F_h is independent of the $x \in \Lambda_{\hat{\mathbf{v}},h}$ that we choose, R and \mathfrak{s} . F_h is the flux of the magnetic field over any surface (or chain) in \mathbb{R}^3 whose boundary is $c(x,\hat{\mathbf{v}})$. We call F_h the magnetic flux on the hole h of K .

THEOREM 6.7. Set $\phi, \psi \in \mathbf{H}_{\langle x \rangle^{4l}}^2(\mathbb{R}^3)^2$ as in Theorem 4.1, with ϕ compactly supported. Suppose that $A_0 = 0$, $B = 0$. For every $A \in \mathcal{A}_{\Phi}^{\text{LR}}(0)$

$$\begin{aligned} \langle US(\underline{A})U^{-1} e^{i\mathbf{x} \cdot v\nu} \phi, e^{i\mathbf{x} \cdot v\nu} \psi \rangle &= \sum_{h \in \mathcal{I}} \left\langle \begin{pmatrix} e^{i(F_h + \Phi_L(A, \hat{\mathbf{v}}))} & 0 \\ 0 & e^{-i(F_h + \Phi_L(A, \hat{\mathbf{v}}))} \end{pmatrix} e^{i\mathbf{x} \cdot v\nu} \chi_{\Lambda_h} \phi, e^{i\mathbf{x} \cdot v\nu} \chi_{\Lambda_h} \psi \right\rangle \\ &+ O\left(\frac{1}{v}\right) \|\phi\|_{\mathbf{H}_{\langle x \rangle^{4l}}^2(\mathbb{R}^3)} \|\psi\|_{\mathbf{H}_{\langle x \rangle^{4l}}^2(\mathbb{R}^3)}. \end{aligned} \quad (6.14)$$

Proof: The result follows from Theorem 4.1, where we take $A^{(1)} \in \mathcal{A}_{\Phi}^{\text{SR}}(0)$ that has compact support (see Remark 3.21 in [8]) so that the error term in Theorem 4.1 is of order $O(\frac{1}{v})$. Furthermore, we take into account that for every $x \in \Lambda_h : \int_{L(x,\hat{\mathbf{v}})} A = F_h - \lim_{s \rightarrow \infty} \int_{\mathfrak{s}(s)} A = F_h + \Phi_L(A, \hat{\mathbf{v}})$

COROLLARY 6.8. Under the conditions of Theorem 6.7, the high-momenta limit (6.14) of $S(\underline{A})$ in a single direction $\hat{\mathbf{v}}$ uniquely determines $\Phi_L(A, \hat{\mathbf{v}})$ and the fluxes $F_h, h \in \mathcal{I}$, modulo 2π .

Proof: The Corollary is a direct consequence of Theorem 6.7.

REMARK 6.9. The results in Theorem 6.7 are also true in the non-relativistic case of the Schrödinger equation in three dimensions considered in [4], with a similar proof, and also in the two dimensional case studied in [7], with the necessary changes due to the differences in the geometry.

7 Conclusions

We prove that the high momenta limit of the scattering operator is given by

$$\begin{pmatrix} e^{i \int_{-\infty}^{\infty} dr (A \cdot \nu - A_0)(\mathbf{x} + r\nu)} & 0 \\ 0 & e^{-i \int_{-\infty}^{\infty} dr (A \cdot \nu + A_0)(\mathbf{x} + r\nu)} \end{pmatrix}, \quad (7.15)$$

where \mathbf{x} is the position operator, from which we recover the electromagnetic field and magnetic fluxes modulo 2π . We prove that $A_\infty(\hat{\mathbf{v}}) + A_\infty(-\hat{\mathbf{v}})$, for every $\hat{\mathbf{v}} \in \mathbb{S}^2$, can be recovered. We, additionally, give a simple formula for the high momenta limit of the scattering operator, assuming the electromagnetic field vanishes (outside the magnet).

The scattering problem that we consider in this paper is important in the context of the Aharonov-Bohm effect [2] (see Section 1.1). The issue at stake is what are the fundamental electromagnetic quantities in quantum physics.

In regard to the description of electrodynamics based on non-integrable phase factors [34] (see also [11]) the physically significant quantities are gauge invariant and (according also to the experiments of Tonomura) the only observable quantities are the electromagnetic fields and fluxes modulo 2π .

Our results show that in the relativistic case (the non-relativistic case is studied in [7]) the scattering operator contains more information than what can be measured in experiments. We can uniquely reconstruct from the scattering operator $A_\infty(\hat{\mathbf{v}}) + A_\infty(-\hat{\mathbf{v}})$, for every $\hat{\mathbf{v}} \in \mathbb{S}^2$, which is not invariant by adding to the flux an integer multiple of 2π and it is not either gauge invariant.

The long range potentials are relevant because a big proportion of the theoretical studies, starting with [2], analyse two dimensional models, in which long-range magnetic potentials are unavoidable. Here we deal with three dimensions, but our results are also valid in two dimensions, with some changes due to the difference in geometry. The two dimensional case was already discussed in [7] (in the non-relativistic case). The results on this work show that our methods in [7] apply to three dimensional models.

References

- [1] R. A. Adams, J. J. F. Fournier, Sobolev Spaces, Amsterdam Academic Press, Oxford, 2003.
- [2] Y. Aharonov, D. Bohm, Significance of electromagnetic potentials in the quantum theory, Phys. Rev. **115** (1959) 485-491.
- [3] S. Arians, Geometric approach to inverse scattering for the Schrödinger's equation with magnetic and electric potentials. J. Math. Phys. **38** (1997) 2761-2773.
- [4] M. Ballesteros, R. Weder, High-velocity estimates for the scattering operator and Aharonov-Bohm effect in three dimensions, Comm. Math. Phys. **285** (2009) 345-398.
- [5] M. Ballesteros, R. Weder, The Aharonov-Bohm effect and Tonomura et al. experiments: Rigorous results, J. Math. Phys. **50** (2009) 122108, 54 pp.
- [6] M. Ballesteros, R. Weder, Aharonov-Bohm effect and high-velocity estimates of solutions to the Schrödinger equation. Comm. Math. Phys. **303** (2011), 175-211.

- [7] M. Ballesteros, R. Weder, High-Velocity Estimates for Schrödinger Operators in Two Dimensions: Long-Range Magnetic Potentials and Time-Dependent Inverse-Scattering. *Reviews in Mathematical Physics* **27** (2015) 1550006, 54 pp.
- [8] M. Ballesteros, R. Weder, Aharonov-Bohm Effect and High-Momenta Inverse-Scattering for the Klein-Gordon Equation, *Annals Henri Poincaré*, DOI: 10.1007/s00023-016-0466-9 (2016) 46 pp, arXiv:1506.01090 [math-ph].
- [9] T. H. Boyer, Darwin-Lagrangian analysis for the interaction of a point charge and a magnet: considerations related to the controversy regarding the Aharonov-Bohm and the Aharonov-Casher phase shifts, *J. Phys. A: Math. Gen.* **39** (2006) 3455-3477.
- [10] A. Caprez, B. Barwick, H. Batelaan, Macroscopic test of the Aharonov-Bohm effect. *Phys. Rev. Lett.* **99** (2007) 210-401.
- [11] P. Dirac, Quantized singularities in the electromagnetic field. *Proc. R. Soc. A* **133** (1931) 60-72.
- [12] V. Enss, W. Jung, Geometrical Approach to Inverse Scattering Appeared in the proceedings of the First MaPhySto Workshop on Inverse Problems, 22-24 April 1999, Aarhus. *MaPhySto Miscellanea* 13 (1999), ISSN 1398-5957.
- [13] V. Enss, R. Weder, The geometrical approach to multidimensional inverse scattering, *J. Math. Phys.* **36** (1995) 3902-3921.
- [14] G. Eskin, H. Isozaki, S. O'Dell, Gauge equivalence and inverse scattering for Aharonov-Bohm effect. *Comm. Partial Differential Equations* **35** (2010) 2164-2194.
- [15] G. Eskin, H. Isozaki, Gauge equivalence and inverse scattering for long-range magnetic potentials. *Russ. J. Math. Phys.* **18** (2011) 54-63.
- [16] G. Eskin, J. Ralston, Gauge equivalence and the inverse spectral problem for the magnetic Schrödinger operator on the torus. *Russ. J. Math. Phys.* **20** (2013) 413-423.
- [17] W. Franz, Elektroneninterferenzen im Magnetfeld, *Verh. D. Phys. Ges. (3)* **20** Nr.2 (1939) 65-66; *Physikalische Berichte*, **21** (1940) 686.
- [18] C. Gérard, Scattering theory for Klein-Gordon equations with non-positive energy. *Ann. Henri Poincaré* **13** (2012) 883-941.
- [19] G. C. Hegerfeldt, J. T. Neumann, The Aharonov-Bohm effect: the role of tunneling and associated forces, *J. Phys. A: Math. Theor.* **41** (2008) 155305, 11pp.
- [20] W. Jung, Geometrical approach to inverse scattering for the Dirac equation. *J. Math. Phys.* **38** (1997), n 39-48.
- [21] F. Nicoleau, An inverse scattering problem with the Aharonov-Bohm effect, *J. Math. Phys.* **41** (2000) 5223-5237.

- [22] S. Olariu, I. I. Popescu, The quantum effects of electromagnetic vortices, *Rev. Modern. Phys.* **57** (1985) 339-436.
- [23] M. Peshkin, A. Tonomura, The Aharonov-Bohm Effect, *Lecture Notes in Phys.* **340**, Springer, Berlin, 1989.
- [24] M. Reed, B. Simon *Methods of Modern Mathematical Physics. I. Functional Analysis*. Second edition. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1980.
- [25] M. Schechter, *Spectra of partial differential operators*. Second edition. North-Holland Series in Applied Mathematics and Mechanics, **14**. North-Holland Publishing Co., Amsterdam, 1986. xiv+310 pp.
- [26] A. Tonomura, T. Matsuda, R. Suzuki, A. Fukuhara; N. Osakabe, H. Umezaki, J. Endo; K. Shinagawa, K. Sugita, H. Fujiwara, Observation of Aharonov-Bohm effect by electron holography . *Phys. Rev. Lett.* **48** (1982) 1443-1446.
- [27] A. Tonomura, N. Osakabe, N. Matsuda, T. Kawasaki, J. Endo, S. Yano, H. Yamada, Evidence for Aharonov-Bohm effect with magnetic field completely shielded from electron wave . *Phys. Rev. Lett.* **56** (1986) 792-795.
- [28] A. Tonomura, F. Nori, Disturbance without the force. *Nature* **452-20** (2008) 298-299.
- [29] A. Tonomura, Direct observation of hitherto unobservable quantum phenomena by using electrons. *Proc. Natl. Acad. Sci. U.S.A.* **102** (2005) 14952-14959.
- [30] R. Weder, The Aharonov-Bohm effect and time-dependent inverse scattering theory, *Inverse Problems* **18** (2002) 1041-1056.
- [31] R. Weder, Selfadjointness and invariance of the essential spectrum for the Klein-Gordon equation. *Helv. Phys. Acta* **50** (1977) 105-115.
- [32] R. Weder, Scattering theory for the Klein-Gordon equation. *J. Funct. Anal.* **27** (1978) 100-117.
- [33] R. Weder, High-Velocity Estimates, Inverse Scattering and Topological Effects. *Spectral Theory and Differential Equations: V. A. Marchenko's 90th Anniversary Collection*. pp. 225-251. Edited by: E. Khruslov, L. Pastur, and D. Shepelsky. American Mathematical Society Translations-Series 2. Advances in the Mathematical Sciences. Volume: 233. AMS, Providence, 2014.
- [34] T.T. Wu, C.N. Yang, Concept of nonintegrable phase factors and global formulation of gauge fields. *Phys. Rev. D* **12** (1975) 3845-3857.

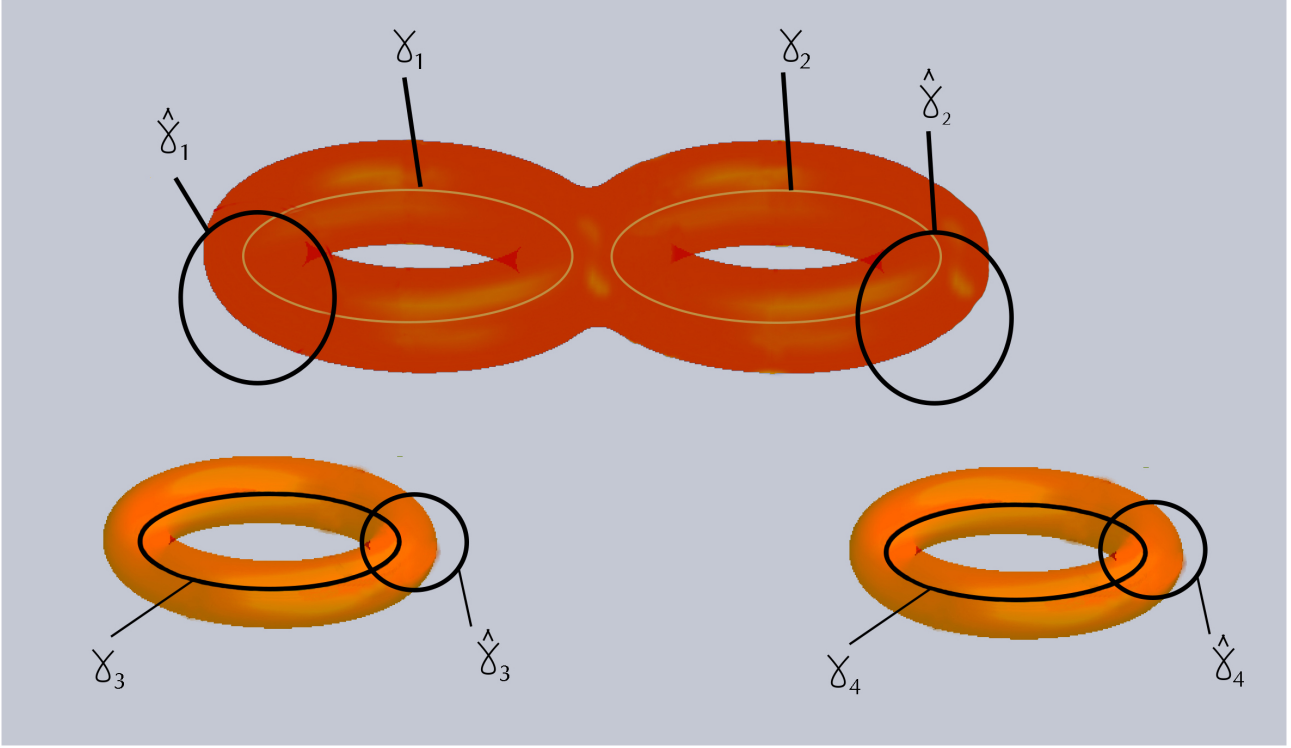


Figure 1: The magnet $K = \cup_{j=1}^L K_j \subset \mathbb{R}^3$ where K_j are handlebodies, for every $j \in \{1, \dots, L\}$. The exterior domain, $\Lambda := \mathbb{R}^3 \setminus K$. The curves $\gamma_k, k = 1, 2, \dots, m$ are a basis of the first singular homology group of K and the curves $\hat{\gamma}_k, k = 1, 2, \dots, m$ are a basis of the first singular homology group of Λ .

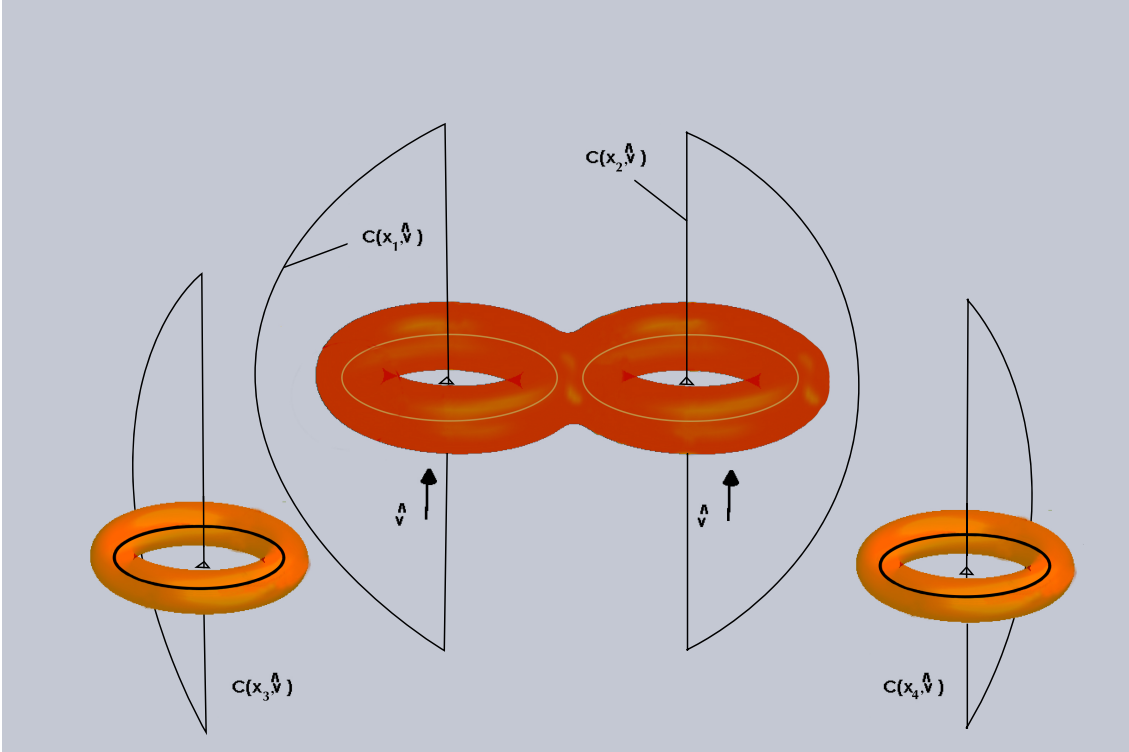


Figure 2: The curves $c(x, \hat{v})$.